WHAT YOUR COLLEAGUES ARE SAYING . . .

"Are you teaching operations, fractions or functions? If so, Harris has some gorgeous ideas for you—showing us the ways they are all 'figure-out-able' with mathematical reasoning."

Jo Boaler
Nomellini-Olivier Professor of Education, Stanford University
Stanford, CA

"*Developing Mathematical Reasoning* is every teacher's guide to breaking away from algorithmic-centered teaching. From the three distortions of mathematics to the hierarchies of mathematical reasoning, Harris helps us understand how math and math teaching have become entangled in a tension between algorithms and reasoning, and then shows us how to untangle this tension through a series of real classroom examples. In so doing, Harris shows us that math is, actually, 'figure-out-able.'"

Peter Liljedahl
Professor of Mathematics Education, Simon Fraser University
Vancouver, Canada

"Harris explores the limitations of an algorithm-centered classroom and emphasizes the need for true mathematical reasoning. By presenting a hierarchy of reasoning domains and advocating for a strategy-centered approach, this book equips educators with vital tools to empower students and deepen their understanding of mathematics."

Graham Fletcher
Math Specialist
Atlanta, GA

"Chock full of real stories about real people engaging with real math, *Developing Mathematical Reasoning* lives up to its title. Harris beautifully empowers educators with practical insights and steps to help students become true mathematical thinkers, not just mimickers—essential for a world that needs confident reasoners."

James Tanton
The Global Math Project
Paradise Valley, AZ

"This book is a gem that should be read by every teacher of mathematics. Harris offers a K–12 continuum of narratives from classrooms and builds a strong argument for why algorithms should not be the focus of instruction if we truly want to produce numerate, mathematically empowered thinkers."

Catherine Fosnot
CEO and President, New Perspectives on Learning
Vero Beach, FL

"This book is a gift for all teachers, especially those of us raised in the era of algorithms and rote memorization. Harris walks you through how to help students reason their way to understand math conceptually. With each step in the progression, you learn how to help students graduate to more sophisticated ways of thinking and math-ing."

Liesl McConchie
Author of *Math With the Brain in Mind*
San Diego, CA

"From the very first page, this book grabbed me and refused to let go. Harris's insights into the challenges of learning mathematics, as well as her joyful explanations of what can be possible when we have the right attitude and mindset, are essential for today's educators to absorb and integrate into their classrooms."

Eddie Woo
Professor of Practice in Mathematics Education, University of Sydney
Sydney, New South Wales, Australia

"Harris critiques traditional math instruction by highlighting three key distortions about what math truly is. She encourages educators to move beyond algorithm repetition and instead promote real mathematical reasoning and problem solving, raising expectations and fostering deeper understanding for all students. A transformative read for anyone looking to elevate math instruction."

Pamela Seda
Founder and CEO, Seda Educational Consulting, LLC
Atlanta, GA

"Harris takes you on an adventure that fast-tracks you along her journey of discovering how students learn best. A must-read for anyone wanting to open students' horizons and get them to use what they already know to tackle new problems."

Christopher Hogbin
Founder, Number Hive
Canberra, ACT, Australia

"This is a timely and, ultimately, brave book about mathematics. Harris shines a light on ineffective practices and reminds us that math is so much more than memorized procedures. Her insights may ruffle some feathers about long held beliefs on math instruction. But the invitation to reach more deeply into real mathematics will open many eyes."

John R. Tapper
CEO & Founder, All Learners Network
Burlington, VT

"This book is a must-have! I grew up in the trap of the algorithm. I made it through school with good math grades because I was a good rule follower. It wasn't until I was getting my master's degree that I learned I didn't know mathematics, I was just good at arithmetic."

Christina Tondevold
The Recovering Traditionalist
Orofino, ID

"Harris is not only a dear friend but also an incredible advocate for teaching math in a way that truly empowers students. In this book, she beautifully abstracts the essence of math, guiding teachers on how to help students deeply understand concepts rather than just memorize procedures. Harris has always been a brilliant resource for educators, helping them uncover the 'why' behind the math, and this book is a testament to her passion and expertise."

India White
TEDxSpeaker, Author, National Ed Consultants
Brooksville, FL

Developing Mathematical Reasoning makes the bold assertion that math instruction should teach students to think mathematically. In this day and age when quick answers are coming quicker than ever, Pamela Weber Harris encourages us to slow down. Using concrete examples and vignettes, Harris demonstrates how traditional teaching methods tend to short-change development by pushing procedural thinking. This book teaches how to navigate around those traps and build classrooms rich with reasoning.

David Woodward
Founder and President, Forefront Education
Boulder, CO

Developing MATHEMATICAL REASONING

Avoiding the Trap of Algorithms

Grades K–12

PAMELA WEBER HARRIS

CAMERON HARRIS, Contributing Writer

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Indexer: Integra
Cover Designer: Gail Buschman
Marketing Manager: Margaret O'Connor

Printed in the United States of America

Paperback ISBN 978-1-0719-4826-2

This book is printed on acid-free paper.

25 26 27 28 29 10 9 8 7 6 5 4 3 2 1

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Videos may also be accessed at **mathisfigureoutable.com/dmrbook**

Preface: Math Is Math. Or at Least It Should Be.

Have you ever had a moment when your understanding of something shifted dramatically? More specifically, when you didn't realize there was even a different way to think? Have you ever rewatched a film or television show for children as an adult, and realized a line or a character or a story means something completely different to you now?

I had that experience with math.

Math is a tricky thing to talk about.

Partially because math is not supposed to be tricky to talk about.

Math is the universal language. The words we use might be different across times and cultures, but the relationships, the equations—those stay the same. Math is supposed to be the unchanging bedrock of science and technology, the one thing everyone can agree on. In any language, $2 + 2 = 4$ is as uncontroversial a statement as it is possible to make.

The language we use to describe math has changed dramatically over the course of human history. The shift from a system like Roman numerals to our modern base ten place system, the invention of zero, irrational, and imaginary numbers, have all dramatically transformed mathematics. Modern mathematicians are still exploring better ways, looking for the next big jump in the way we describe and do math. And that's all beyond the scope of this conversation.

As uncounted numbers of frustrated parents have exclaimed, "You can't change math! Math is math!"

Unfortunately, that is only correct if what we learned as math *actually was math*.

What if math is actually something different than what many of us thought? What if it has actually been obscured, misrepresented, and hidden behind shortcuts and tricks until it is practically surrounded by half-truths, overgeneralizations, accidental lies, and unconnected trivia, such that our perspective of math can be so damaged we don't understand what math even is? And then we teach the next generation and pass on those misconceptions, and they teach the next, and so on. It's like a bad game of telephone, except the original message was wrong to begin with.

I've found that this miscommunication spiral has resulted broadly in three distinct distortions that affect how many people view mathematics and the teaching of it. The result is that even though there are very different ways people all over the world perceive math, many don't realize there is another way, let alone multiple ways.

Note that these distortions are not binary. Through their life experiences some people are dealing with more distorted math than others, while still others might be dealing with very little distortion at all. However, my experience in 40 years of teaching suggests that the vast majority of people, students or not, are dealing profoundly with one of the following three.

THE THREE DISTORTIONS

THE FIRST DISTORTION: MATH IS NOT FIGURE-OUT-ABLE, IT'S ROTE-MEMORIZABLE

"Are these the problems we did on Tuesday, or the ones from Wednesday?"

"Is this where we cross multiply and divide or find a common denominator? Or cross cancel?"

"I've got this far. What's the next step?"

"6 times 8, like the garden gate, is made of sticks so it's 56. . . . Wait. . . ."

"If I put my fingers up like this, then it's these fingers and those, so 9 × 7 is 6 and 3, 63."

"Does slope go here in the formula? Do I have the right formula?"

"Since math is about memorizing all of these things, I will help students memorize them with mnemonics, stories, and songs."

Under this first distortion, mathematics is an arbitrary set of rules and procedures (such as algorithms). To do well, students must decide what to do, in the right order, and copy the teacher's examples. The why, the background, or the connection to other mathematics doesn't make a difference in getting right answers, so that discussion is irrelevant. "Please just tell me what to do and let me get my homework done." This distortion holds that math has little to nothing to do with your life's experiences.

Some of you read this and nod, "Yes, that's what math is. If my students do this, they will be successful. If they don't have good memories, they won't do well. If they don't do well, they might have math anxiety, they won't pursue STEM fields. That's just how the world works."

This was me.

To be clear, I was *good* at this conception of math. I didn't become a high school math teacher because I struggled with the subject. I *excelled* at thinking about math this way. I *excelled* at teaching it.

Then I found a better way. To my delight (once I got over the existential crisis), I discovered that students struggling with memory can be reached. That students who appear unwilling to exert themselves will self-motivate if the distortions are removed. That students not struggling to mimic can dive deeper and soar higher.

THE SECOND DISTORTION: MATH IS FIGURE-OUT-ABLE FOR ME, BUT NOT FOR EVERYONE

> *"I mean, you could do all of those steps, but $99 + 47$ is just like $100 + 46$."*
>
> *"Why would you go to all that effort? Plainly, $\frac{1}{2}$ of $\frac{3}{5}$ is just $\frac{1.5}{5}$, so $\frac{3}{10}$."*
>
> *"Why do I have to show those steps? It's just obvious."*
>
> *"I'm not sure why no one else is just figuring these out. Maybe they need the steps."*
>
> *"I guess I have the math gene. I don't know how to help people without the math gene think like I do."*
>
> *"I do things in my head, but I know I'm supposed to teach the rules and steps. That's what you do."*

Under the second distortion, math is figure-out-able, but for some reason not for everyone. Math-ing means to use what they know to reason about new things. Someone who is under the spell of the second distortion believes that "I can add to my repertoire and keep building because it all makes sense. I do not have to wait until someone shows me a rule. But for some reason, other people can't or won't."

Many with this distortion were taught by well-meaning people with the first distortion. And they thought that teaching was ridiculous, nonsensically inefficient, or at best not needed. They watched teachers conduct sing-alongs and teach rhymes to memorize algorithms, wondering all the time what in the world any of this had to do with math.

This was my eldest son's experience, who upon being taught a subtraction algorithm in first grade thought it was unnecessarily complicated and invented his own. This is using $7 \times 7 = 49$ to reason that $8 \times 7 = 49 + 7 = 56$. This is reasoning about the equation of a new function using the equation of another function without starting from scratch with a formula.

To fit the definition of this distortion, this reasoning ability is gained *in spite* of how someone was taught, not *because* of it.

The problems this distortion cause become most obvious when it is time for this person to turn around and *teach* math. They learned real math, but didn't realize it was in spite of how they were taught, not because of it. They teach their students the same way they were taught. Some students "get it" the way they did, but most don't. Many assume at this point the difference must be that of innate ability, that the students who get it have "the math gene," and those without, don't.

Sometimes when I discuss these three distortions, people say, "I want to have the second distortion." But remember, this is a distortion because people under this distortion do not realize they can purposefully teach what they actually did naturally while their teacher was drilling step-by-step procedures. Of course, we all wish we had the natural talent to recognize mathematical patterns without being intentionally taught them, but we can't invent natural talent. What we can do is teach the real math-ing to students instead of the fake math of memorizing and mimicking.

THE THIRD DISTORTION: MATH IS FIGURE-OUT-ABLE, BUT NOT FOR ME

> *"I still don't understand. I think I should be able to understand, but this seems really random."*
>
> *"Yes, I could do what you're telling me. But no, I don't want to just do the steps because it doesn't make sense."*

> *"I mean, I could just try to memorize and do what you're saying, but I know I'll mix it up because I don't get it."*
>
> *"Math is hard to understand. I'll do my best to clearly explain the parts I get and be patient to explain as many times as needed."*

These people think math should make sense. They should be able to figure out what to do because memorizing what is arbitrary doesn't work for them. They have a sense that *math* is not arbitrary but, for whatever reason, don't make the connections the same way people under the second distortion do. Some students under this third distortion believe their teachers are deliberately holding back, deliberately not explaining the math.

Because math is so often presented as something to rote-memorize, these students are left to figure it out on their own. These students often abandon their reasoning when it doesn't match what the teacher is doing, assuming that reasoning must be wrong. The algorithms work against their intuition, invalidating their thinking.

Rote-memorize: to commit a fact to memory independent of the surrounding context that explains why that fact is true. For example, memorizing the names of capital cities with flash cards or memorizing multiplication tables without building the accompanying Multiplicative Reasoning that explains and justifies why the multiplication table is the way it is.

Math starts looking like a bad magic trick. Any piece of math—subtraction, the Pythagorean Theorem, pi, the slope-intercept equation—all might as well be ink blots to memorize, because none of them make any sense.

The rest of the book will illustrate how algorithms work against intuition.

These students are stymied when $7 \times 8 = 56$ makes as much internal sense as pineapple times automobile equals tiger. They try to memorize the songs and pictures and sayings, but they know they're not relevant. Knowing $7 \times 7 = 49$ is of no help learning $7 \times 8 = 56$, because there is no connection between pineapple times automobile equals tiger and pineapple times airplane equals lion.

Many, many people I've talked to about "memorizing their multiplication tables" saw them exactly this ridiculously. Disconnected and meaningless.

Given that reality, many under this distortion disengage from the learning as a defense mechanism. They invest less emotionally, to soften the repeated shaming that not understanding brings. They succumb to "Just tell me how to do it; I'll fail at that, and we can move on."

The truth is that often their teachers are operating under one of these three distortions. Individuals working under the first distortion don't know math is figure-out-able, and so understandably won't teach it that way. Those under the second distortion know math is figure-out-able, but don't know how to teach it that way. Finally, those dealing with the third distortion never felt like they succeeded learning math themselves. They teach the way they were taught, hoping their students will figure it out where they didn't. Usually while stressing about it. A lot.

TRY IT

Consider which of these distortions resonates with you or with your experience as a student. Consider how the lens you've had may have colored your view of mathematics or mathematics teaching.

THE REALITY: MATH IS FIGURE-OUT-ABLE FOR EVERYONE

Real math, math-ing, is not trivial. It is not obvious. It is not simple. But it can be taught.

In my 30 years in math education, the one truth that has been reinforced over and over again is that everyone can do more real math than fake math. Everyone can do more math when that math is built on what they already know rather than shoehorned on the backs of contextless rote-memorization. In other words, everyone can *math*. Everyone can have their horizons open up and have more choices.

What does it mean to "math"? Real math, doing real mathematics, begs a verb like math-ing. Deborah Crayton has coined the term math-er. "Readers read. Writers write. Mathers math" (Crayton, 2026). Cathy Fosnot uses mathematizing (Fosnot & Dolk, 2001, p. 4). Math-ing or mathematizing as a verb describes the mental actions that mathematicians do. See Chapter 1 for more about what this means.

The first distortion is inherently limiting. The mountain of facts and steps to memorize become too much. Learners can't keep it all straight or use any of it to reason about new things. For many, this happens as they move into long division, fractions, or algebra.

People under the second distortion usually make it the farthest. But how much more could they have learned faster if their growth were assisted by their teacher, instead of having to figure it out on their own? How many more people could join them in these STEM fields if they were actually taught real math-ing?

When helped to *math* in a real way, people who were under the third distortion gain the confidence to invest emotionally again because their effort is rewarded. Frustration and anxiety vanish into comprehension and proficiency. They know they can understand, and indeed they do.

In university classes and in-service workshops I lead, when I get people math-ing, many for the first time, I get these reactions:

- First distortion: Whoa, I did that. That was *my* thinking. Wait, we can teach math this way?
- Second distortion: Yes, that's what I've been doing in my head, but now I'm seeing that I can teach kids to do what I've been doing. Cool.
- Third distortion: Hallelujah—I knew I could understand! Now I can help my kids *math* with understanding too.

TRY IT

By acknowledging the way you viewed the nature of doing and teaching mathematics, you can choose today to align your teaching with what you actually believe.

Take this quiz online:

https://qrs.ly/x1g41d1

To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

It is reproduced on the next page.

The Perspective Quiz

What did you think it means to do and teach mathematics?

When you think about learning mathematics as a child, do you . . .

A. Break out in a sweat, get nervous, and wonder if you'll remember how to do the problems?

B. Smile, remembering some fun problems you worked on and patterns you found that helped you make sense of problems?

C. Feel cheated, like you know you could have learned more and done better if your teachers would have explained more, or better?

D. Remember knowing that if you just practiced a lot, you could remember what to do when?

If your childhood friend had asked you a mathematics question, you would have . . .

A. Clearly told them the rule and the steps to do the problems or looked it up in a textbook or online help.

B. Told them they will have to ask someone else because you never did understand how to do those problems.

C. Looked at the problems to see what relationships you could use to solve them, but then showed your friend the steps you learned in school.

If you had missed a day of school, you would have . . .

A. Looked at the missed assignment, confident that you could probably figure out the answers to the problems by thinking logically about them.

B. Waited until the teacher showed you the rule and the steps to solve the problems.

C. Tried to figure out how to solve the problems, but if you couldn't readily, asked the teacher to explain what was happening and why, so that you could understand how to solve the problems.

If you didn't understand a teacher's explanation, you thought that . . .

A. It might be your fault, but maybe you just were not a math person.

B. You could sit with the problems, think about them, and figure out a way to make sense of them.

C. You needed to see the steps again and practice some more.

When a teacher began to explain the lesson for the day, you hoped . . .

A. The teacher would help you understand what it all meant and why because you knew then you had a chance of getting it right. If you didn't understand, you might get correct answers today, but you wouldn't be able to hang on to them without understanding.

B. The teacher would just tell you what to do and how. "Give me the steps and let me practice them. Please don't tell me why or give me more than one way."

C. You'd have a chance to play with the concept, the numbers. You wanted to try your hand at solving the problems on your own. If the teacher made you mimic their steps but you didn't need to, that was frustrating.

A teacher said, "Show me your work." What you heard was . . .

A. Use what you know, how you understand what's going on, to make sense of the problem. Then write something on paper, probably what the teacher had shown, because you may not know how to write down what was happening in your head.

B. Copy exactly the steps that you were shown in the right order. Practicing the correct steps in the correct order is the work—that's what it means to do mathematics.

C. Show me that you understand what the teacher was asking. If you did not understand, you didn't want to just do what the teacher had shown because then you knew you wouldn't be able to do it again. It didn't make enough sense for you to own it.

FREQUENTLY ASKED QUESTIONS

Q: What if none of these distortions feels like they apply to me?

A: First, remember that these descriptions are generalizations and not meant to describe any one person exactly. Second, many people feel like they've changed at some point. They feel like they started out believing that math made sense and that they were capable of reasoning through it and then later found themselves stymied. For example, *This year's math isn't making sense like last year's*

(Continued)

(Continued)

did, or *I thought I was a math person, but I guess not.* Many students begin their schooling with a clearer idea of what math is than they have by the end of their first multiplication/fraction/long division unit. They then end up under the first and second distortions. You could use this discussion to understand why this global conversation about mathematics education is so complicated, tricky, and subtle, because many are coming from these different distortions. The most important part of this discussion is to point us all to real math-ing. The three distortions are frameworks that suggest why people might disagree about how mathematics should be best taught.

Q: What if my teachers taught me conceptually, and now I teach conceptually too? I don't seem to fit your three scenarios.

A: Give the rest of this book a read. If you get to the end, and sure enough, your teachers and you approach teaching mathematics as developing a hierarchy of reasonings that don't rely on mimicking any algorithms *and* that every single one of your students can learn math this way—then, fantastic. You are one of the lucky few. Otherwise, you might be dealing with the second distortion.

ABOUT THIS BOOK

Chapter 1 lays out how the implementation of algorithm-centered math education today, its methodology and goals, are often at cross purposes with the true nature of mathematics and doing mathematics. It then goes deeper into how the proliferation of algorithm-centered teaching is largely responsible for these issues, and that understanding those issues presents the best opportunity for improving math education. When we understand the nature of mathematics, we can mentor students to *math* like mathematicians.

Chapter 2 introduces the hierarchy of mathematical reasonings essential to the learning and progress of all mathematics students. It goes through the major domains of reasoning: Counting, Additive, Multiplicative, Proportional, and Functional, laying out how they build off each other and represent tiers of increasingly sophisticated thinking. It introduces *sophistication* as a descriptor of the thought processes used when solving math problems. This term is needed because although it includes ideas of speed and efficiency, it also includes the magnitude and complexity of mathematical relationships in use, which neither speed nor efficiency denote. The chapter will go into this topic in far more depth, but I want to note here that the term *sophistication* as it is used in this book is never a value judgment of a person or their thoughts. It is only used as a relative measuring stick to place where a given method of solving a problem falls on the growth continuum. A student currently developing in the Counting Strategies domain of reasoning is not better or more valuable than a student developing Functional Reasoning. They are simply at different stages of development.

Chapters 3–6 each define and illustrate one reasoning domain and how rote-memorizing and mimicking algorithms can trap students into using less sophisticated reasoning than the problems call for, therefore limiting students' reasoning growth. Each includes a detailed, step-by-step walkthrough of at least one commonly used algorithm and an explanation of how at each step it can undermine students' opportunity to grow their mathematical reasoning ability. These chapters illustrate the major mathematical strategies to develop in place of those algorithms and discuss the advantages the strategy-centered approach brings. These advantages include an often faster, almost always longer lasting, and more complete understanding of content.

As used in this book, the term *mathematical reasoning* does not mean just a general ability to think. This is not a fuzzy, "think better" approach that doesn't include doing the math and getting results. Mathematical reasoning is about building stronger brains and expects more, not less, from students, giving them the tools to actually be successful at math-ing. It demands increasing sophistication of strategy. This means meeting students where they are, and then helping them develop from there. For example, students will not only know their multiplication facts, they will actually own them and be able to use the relationships in problems. It includes content-specific milestones such as understanding of integer addition and subtraction, multiplication of fractions, and so forth.

The final chapter, Chapter 7, answers the question, If mathematics teaching is not all about repeating the steps of algorithms, then what is it? The chapter outlines steps teachers can take to improve their own and their students' mathematical reasoning ability regardless of their current reasoning level or what content they need to teach.

For the content you teach, you can work to solve problems using what you know and learn the major models and strategies for that content. You can work to elicit and represent student thinking, making thinking visible, point-at-able, and discussable. Lastly, you can work on high-leverage teacher moves and sequencing tasks, with an eye toward moving the math forward and meeting all students' needs.

Each chapter includes tips and FAQs throughout, as well as actions the reader can take—either personal exercises or things to try in class.

Corwin and I will be publishing four additional grade-specific companion books (K–2, 3–5, 6–8, and 9–12) on a six-month cadence once this book is released, which will offer more ideas, more practice, and more practical advice, concentrated specifically on each grade band. These books will be complementary to this anchor volume, which we believe is necessary to set the foundation of the discussion on developing mathematical reasoning.

FOUNDATIONS

All of the ideas, concepts, methods, and proposals for how to teach more students more math contained in this book have their foundation in 30 years of study and classroom-based research. *Development of Mathematical Reasoning* is the result of synthesizing research and personal experimentation with teachers and students in real classrooms to find what works and what doesn't—what cultivates real understanding versus what gets quick answers at the cost of long-term development.

My work is influenced by that of Fosnot and Dolk (2001) in their *Young Mathematicians at Work* series and Fosnot's *Contexts for Learning*, which showed me children reasoning about content and how to get them to do it; Jean Piaget (1896–1980), the founder of cognitive development; Hans Freudenthal and the Freudenthal Institute in the Netherlands and their Realistic Mathematics Education philosophy; Constance Kamii and Ann Dominick, who published "The Harmful Effects of Algorithms" in 1998; and Liping Ma, who coined the PUFM "profound understanding of fundamental mathematics" in her *Knowing and Teaching Elementary Mathematics* (2010). Other noteworthy influences are the work of Marilyn Burns, *Math Recovery*; Les Steffe, Anderson Norton, Amy Hackenberg, and Susan Lamon's *Teaching Fractions and Ratios for Understanding*; NCTM's *The Teaching and Learning of Algorithms in School Mathematics* (1998); *Developing Mathematical Reasoning in Grades K–12* (1999); Kazemi and Hintz, in their Intentional Talk (2014); Smith and Stein (2011) in their *Five Practices for Orchestrating Productive Mathematics Discussions* the textbook *Functions Modeling Change* by Connally et al. (2000); and recently *Building Thinking Classrooms* by Liljedahl (2021) and *Rethinking Disability and Mathematics* by Lambert (2024).

We already have many of the foundational elements of teaching math better. The last four major standard shifts in the United States—"Professional Standards for Teaching Mathematics" (NCTM, 1991), "Principles and Standards for

School Mathematics" (NCTM, 2000), the "Curriculum Focal Points" (NCTM, 2006), and the Common Core State Standards (2010) are important moments in recent history that each tried to delineate what should be taught when in school mathematics. The move toward developmental progressions based on research was necessary and helpful.

Simultaneously, the National Science Foundation funded several universities to create textbooks that were more aligned with that current thinking. Textbook series like *Discovering Mathematics; Investigations in Data, Number, & Space; Math in Context;* CMP (the Connected Math Project), *Everyday Mathematics, Math in Context, CORE Plus*, and COMAP had many schools and teachers trying to get students to investigate and discover math that was more in context, using manipulatives, models, and technology. Many teachers, who were like the earlier me, tried these innovative approaches but didn't understand why it was necessary or what the goal even was. Most importantly, the new standards and the textbooks based on them do not account for the heavy distortions that most teachers and students operate under about the nature of math—for example, the myth that one must have the math gene to excel at math or that math must be memorized and mimicked.

This book brings together the outstanding research that exists and the understanding of the way it has been misunderstood to help leaders and teachers navigate where to go now.

A word about research.

It is frankly *easy* to take two groups of students for a few weeks, drill one group and not the other, and then show that the drilled group "knows" more. Research like this rarely gives any indication of how long students will retain what was drilled, or if the isolated drilled "knowledge" is weaving well (or at all) into the interconnected web of mathematical knowledge needed to further support learning. I find this research unhelpful.

The research that I find much more useful consists of those studies where researchers create tasks, facilitate them with students, learn more about how students learn and the interconnectedness of the mathematics, tweak based on the results, share what they have learned, and then rinse and repeat. These more useful studies show what students know long after the initial teaching and how it connects to and supports future learning. This research helps me as a mathematician, a mathematics teacher educator, and as a teacher myself to better understand *mathematics for teaching* (Ball, Thames, & Phelps, 2008), and how to help teachers and students develop as genuine doers of mathematics.

HOW TO USE THIS BOOK

This book is meant to be read from beginning to end, at least on the first read-through. You may be tempted to skip to what looks like your grade level, but *an essential part of avoiding the traps of algorithms is understanding the prior context of what comes before your grade level*. Once you've gotten to your grade level, don't stop there: Understanding how your content affects reasoning in the latter grades is also essential to developing mathematical reasoning.

There are frequent problem-solving examples throughout the book. When you reach one, pause! Think. Solve. Think about your thinking. Examining your own thought processes and pondering how your students would react to these problem-solving opportunities is a crucial part of making the most of this book. Fundamental to the Math Is Figure-Out-Able philosophy is that we have to *math* to learn how to teach math better. You must establish the relationship in your own head so you have something to hang other people's thinking on.

After a first reading, teachers seeking to hone their skills and understanding about their specific subject areas would do well to study the chapters covering those topics. Coaches or leaders may want to read the whole book multiple times—first from a learner's, then a teacher's, then a leader's perspective.

In this book, I use teacher's and student's names where I have permission and pseudonyms where I do not. I am so grateful for the expert teachers who allowed me to work with them and their students.

Discussion Questions

1. When you think of math as a verb, *math-ing*, do you think more of reasoning, creating arguments, justifying, critiquing, or do you think more of rote-memorizing and mimicking?
2. Do you recall being under any of the three distortions? What sparks for you when you read these descriptions? Is anything missing? How would you tweak them?
3. Might your colleagues be under any of these distortions? How do you know?
4. What are you wondering about as you finish reading this preface?

Acknowledgments

To my son Cameron, a mighty thanks for being the kid who blew my mind over and over again, giving me the gift of real math-ing. And how fun is it that you helped me write this book?! Without you, I don't think I would have gotten this done. Definitely not with as much humor.

Thanks to my other children, Matthew, Craig, and Abigail, who were the best guinea pigs, and my husband, Daniel, for everything, including helping me make the sticky bits less sticky. And Abby—great job on the graphics!

Thank you to Kim, Sue, and the rest of the gang at Math Is Figure-Out-Able. And thanks to Ann Latham for her meticulous work with permissions.

Thank you to Garland Linkenhoger, the Charles A Dana Center at the University of Texas, T3-Teachers Teaching with Technology, Paul Kennedy, Cathy Fosnot, Scott Hendrickson, Kathy Hale, and all my colleagues who have shaped my teaching career.

Thank you to Debbie Hardin, Senior Acquisitions Editor at Corwin, who has walked me through the whole book-writing process with a skilled hand. And thank you to everyone at Corwin for giving me the opportunity to get this message to the world.

Here's a respectful and grateful thank you to the thousands of teachers who have been with me on my journey, taken our workshops—online and in person—and allowed me into your classrooms. You work tirelessly to improve the lives of your students. It shows. I hope this book does justice to your journey.

And to the Author of life, thank you God for giving me a message worth sharing.

PUBLISHER'S ACKNOWLEDGMENTS

Corwin gratefully acknowledges the contributions of the following reviewers:

Brandon Pelter
Math Teacher, Bridgeport Public Schools
Norwalk, CT

Kimberly Rimbey
Chief Learning Officer and CEO, KP Mathematics &
Kim Rimbey Consulting
Author, Corwin
Phoenix, AZ

Adina Rochkind
5–8 Math Curriculum Coordinator
Baltimore, MD

Brendan Scribner
Math Consultant, Scrib Mather, LLC
White River Junction, VT

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About the Author

Pamela Weber Harris is changing the way we view and teach mathematics. Pam is the author of several books, including the *Numeracy Problems Strings K–5* series, *Building Powerful Numeracy,* and the *Foundations for Strategies* series. As a mom, a former high school math teacher, university lecturer, and an author, she believes everyone can do more math when it is based in reasoning rather than rote-memorizing or mimicking. Pam has created online *Building Powerful Mathematics* workshops and presents frequently at national and international conferences. Her particular interests include teaching real math, building powerful numeracy, sequencing Rich Tasks to construct mathematics, using technology appropriately, and facilitating smart assessment and vertical connectivity in curricula in schools PK–12. Pam helps leaders and teachers make the shift that supports students to learn real math because math is figure-out-able!

CHAPTER 1

Math Is Figure-Out-Able

"Anyone like gum?" I ask a class of 28 high school seniors. We are about to video classroom interactions to put in my *Developing Mathematical Reasoning* online workshop. This is the warm-up the day before, to let me get to know the students, practice pronouncing names correctly, and give students a chance to know what to expect when the film crew arrives the next day.

I start with a *Problem String*, an instructional routine designed to build a specific mathematical model, strategy, or concept.

These students are taking a course called Advanced Quantitative Reasoning, an alternative to calculus. This likely means these students had been fairly successful in their previous courses, algebra 2 and precalculus, but did not choose to take calculus for their last year of high school math.

"If 1 pack of gum has 27 sticks, how many sticks are in 10 packs?" I ask.

I don't wait very long for this answer. Maddie replies, "270 sticks."

"How about 9 packs? How many sticks in 9 packs?" I ask.

After some think time, Kayla replies, "It's 243 sticks. Just take away 27 sticks from 270."

"How did you do that subtraction?" I ask.

"Take away 30 and give back 3," she says. I represent that strategy on the board, and we briefly discuss it and two other ways students were reasoning about 270 – 27.

I follow with, "How many sticks in 100 packs?" and Victor answers quickly with 2700 sticks.

When I ask, “What about 99 packs?” students smile. Not a common experience for many teachers when asking 99 times anything.

Mia answers, “Just take away 1 pack. So, 2700 – 27. That’s 2673.”

After a couple of students share how they reasoned through the subtraction 2700 – 27, Cameron, sitting right in the middle of the class, looking very thoughtful, raises his hand.

“It’s almost”—Cameron pauses—“it’s almost like you want us to use *what we know* to solve these problems.”

“Yes,” I smile. “Yes, I do.”

Every other student in the room nods thoughtfully, like this is new and noteworthy. A realization that is both wonderful and tragic. Wonderful, because these students have just experienced what it means to really do math. Tragic, because experiences like this should have defined their entire math education, not just be featured as a last-minute footnote.

“Keep using what you know,” I continue. “What if we had 2646 sticks of gum—how many packs?”

Students look at what is being represented on the board and use that to reason that 2646 is 54 sticks, or 2 packs, less than 100 packs. These students have confidently reasoned that 2646 sticks are in 98 packs, or $2646 \div 27 = 98$, without steps, without mimicking.

What students are doing is not a bunch of random mental math tricks. These students are developing one of the major strategies that spans the operations and makes use of the distributive property—the Over strategy. In this case they are multiplying by a bit too much and adjusting back.

It's focused, purposeful instruction in real math-ing.

Is this what math class looked like and felt like when you were a student?

I am not advocating that we stop teaching grade-level content, making seniors back up to multiplication in isolation. This particular Problem String is so good because we can get students reasoning about multiplication, work on Additive Reasoning with subtraction, and simultaneously build Proportional Reasoning because we're using a ratio table as a model. We can then extend that String to graphing those ordered pairs, writing the function to match the data, $y = 27x$, *because the number of sticks equals 27 sticks per pack times the number of packs. We can also discuss the graphs of related transformed functions, like the line containing (1, 28) and (2, 55) or the line containing (1, 26) and (2, 53). There is a lot of meat here, real content with many access points to allow students to enter the problems and also to challenge all students. You can find more examples of Problem Strings that build numeracy into middle and high school content in my* Building Powerful Numeracy for Middle and High School Students *(Harris, 2011) and* Lessons & Activities for Building Powerful Numeracy *(Harris, 2014). Also watch for grade-specific companion volumes I am publishing with Corwin (K–2, 3–5, 6–8, and 9–12), which I am publishing in order every six months, starting with the release of this anchor volume.*

MATH IS ACTUALLY FIGURE-OUT-ABLE

Worthwhile mathematics teaching is about helping students to use what they know to reason through problems, strengthening their minds as they grapple with and make sense of increasingly complex mathematics.

The purpose of mathematics instruction is to build students who *math* (Crayton, 2026), not students who are mimicking or randomly guessing. Teaching mathematics, mentoring students to *mathematize* (Fosnot & Dolk, 2001a), requires focused direction and pedagogical skill that capitalize on students' existing mathematical knowledge and intuition and guide them to develop ever more sophisticated powers of reasoning.

Math is not rote-memorizable; math is not random-guessable. Math is figure-out-able.

SWIMMING WITHOUT WATER

Many current math classes operate like learning to swim without water.

To learn to swim, a person gets in the water. They begin by doggy paddling when they can just barely reach the bottom. As they learn strokes, become more confident, and build stamina, they swim farther and in deeper water. In many math classes, it's as if students are watching from outside the pool, observing people swim and mimicking what they can see above the water, but they actually have no idea what's going on under the water. Students are not privy to the thinking going on in a mathematician's mind, the choices made along the way, the start and stop, try, fail, and correct that happens when doing something new. Students are told they're swimming as they sit on the edge with feet dangling in the water, but they never actually learn to swim.

When students in math classes rote-memorize facts in isolation and mimic procedures, they're being told they've learned more and more math, but in reality their brains never get any stronger mathematically. Their mathematical reasoning ability isn't growing, only their catalog of memorized facts. These facts are essential, but they are not sufficient.

Mathematical reasoning in this book is not some generalized problem-solving schema, some fuzzy thinking better. *Mathematical reasoning includes content—this means that a student reasoning mathematically is using mathematical relationships, properties, and models to argue, operate, and solve problems and in the process learn more mathematics content. You will learn more about this in Chapter 2.*

Students are told they are doing math as they mimic procedures, carry the 1, cross out the 0, keep change, flip. But they aren't encouraged to see the beauty of a well-formed argument, a clever strategy, or a model that illuminates and helps them own interconnected relationships. When math students don't know what it means to critique reasoning, logically prove generalizations, and use what they know using mathematical relationships, they are not experiencing real math-ing.

Many of us were trained to teach mathematics as rote-memorizing steps and mimicking actions without ever engaging in the mental actions that mathematicians use. This is the way most of us learned, ourselves. But as Maya Angelou said, "When we know better, we do better." In the sticks of gum Problem String, those students were using what they knew and reasoning with mathematical relationships, not parroting procedures. Procedural mimicry is not doing real math—it's swimming without water. Many of us have been trapped over the years into thinking that we are doing the work of mathematics, when in reality we aren't math-ing at all. *Mimicking squelches opportunities to develop more sophisticated thinking, the doing of real math.*

TRY IT

Consider ways that you have inadvertently done the heavy lifting on your students' behalf, or allowed them to push down the metaphorical piano keys but kept them from doing the real work of math.

To be clear, I am not blaming individual teachers or schools. I am *certainly* not blaming you. For heaven's sake, I was the teacher who was trying to get students to swim when I didn't even know what swimming *was*. I used rhymes and sound effects (literally) when my students could have been producing actual music. This book is my attempt to help you learn what I did, to have the epiphanies I did, without the years of research and experimentation it took me.

This isn't about blame or shame. It's about helping you do and teach more real math.

When I say math like a mathematician, *I don't mean the math-ing that mathematicians do as full-fledged adult mathematicians; I mean the way mathematicians mathematized in first grade, fifth grade, eighth grade. Over lunch in the dining hall in the Queen's College at*

Oxford, several mathematicians who had just heard my presentation at the Mathematics Education for the Future Project conference told me story after story about how they just ignored the steps their teachers showed them. They chuckled as they asked me, "Why would you do all those steps when you can just reason through the problems?"

OUR BIGGEST OPPORTUNITY FOR IMPROVEMENT

Many math education articles, books, and talks begin with a montage of depressing statistics about math achievement—more specifically, the lack thereof, the dwindling pool of students with the math qualifications to follow STEM careers, and so on. Most conclude with some flavor of insistence that this woeful situation is because our math students' speed is not speedy enough, and their accuracy is not accurate enough, then propose ways to fix those problems. Many of the proposed solutions describe where teaching should fall on the continuum between extreme versions of direct teaching and inquiry. One extreme proposes demonstrating to students exactly how to solve all the types of problems they will need to solve in a particular class. The other argues that we should give kids prompts and manipulatives and let them explore and discover how to solve problems on their own.

Some blame direct teaching for poor student performance: "Stop doing that," they say. "Do discovery, inquiry learning instead. It's more engaging, fun, and real world when students do problem-based learning. Get kids some conceptual understanding so they'll be able to do the algorithms better." These teachers might hope that the right combination of exploration and engagement will help students understand and learn the algorithm better.

Others claim the opposite, that not enough direct instruction is happening in the classroom: "Stop trying to get them to discover everything—that is fuzzy math, where teachers are making students guess at and reinvent math. Why make kids struggle? Just tell them how to do it clearly. Have kids memorize the basic steps so they can build the more complex math on that foundation." Teachers on the far end of this spectrum might teach math the same way they would teach capital cities in a geography class—with flash cards, rhymes, and mnemonic tricks, as if there was no underlying logic behind the answers. They I-do-we-do-you-do their way through a chapter in a textbook, demonstrating algorithms and watching to make sure students can do all the steps correctly.

Note that neither party is discussing inventing or leveraging the power of algorithms in computing. Algorithms are at their best when they are invented by humans, but carried out by computers. The only valid reasons for a human to follow the steps of an algorithm by hand revolve around figuring out how one works so a variation can be created, or to discover why an algorithm is not producing the desired result. Neither of those have anything to do with finding the specific answer to a specific problem, which is always how they are employed in a K–12 math classroom. Algorithms are amazingly powerful mathematical inventions that are essential to much of modern technology. That same amazing, ruthless generality and applicability are what make them lethal to a learning environment.

These two approaches feel like opposites, but they share the same misguided goal: that students should get answers easily and quickly and that the best, most efficient way to do that is by repeating algorithms. Procedural fluency, understood as easily performing the steps, is the hallmark. Either way, repeating the steps of an often inefficient, nearly always opaque, mind-numbing algorithm is put on a pedestal as the supreme example of what mathematics competency looks like. Where they differ is how to teach those algorithms.

I am using terms to describe some muddy philosophies: direct, explicit teaching and discovery, inquiry teaching. To make my arguments clear, I am caricaturing both, describing the extreme ends of the spectrum. The reality is that most often the argument between the two is an unhelpful false dichotomy, particularly when algorithms are the goal. I argue we need to change the goal. Then the whole conversation shifts.

Let's follow this to its logical conclusion. If the goal of math class is to produce students who produce correct answers, mathematics education could be reduced to a few weeks covering how to use generative AI.

Math class in the 21st century cannot be about answer-getting (Daro, 2014).

Once we recognize that producing answer-getters cannot be the point of math class, teaching algorithms loses much of its justification. If we change the goal of math class, algorithms as teaching tools are no longer the focus.

Our greatest opportunity for improvement lies in removing algorithms from the goal of math class. Believing that doing math means memorizing and mimicking algorithms, even with understanding, is the same as believing that writing the steps of a workout would ever be sufficient for getting in shape.

That watching someone riding a bike will provide the experiential balance and coordination necessary to actually ride the bike yourself.

Our greatest opportunity for improvement lies in removing algorithms from the goal of math class.

Removing algorithms as teaching tools in math education will not automatically fix all of mathematics education, any more than removing a splinter will heal a wounded foot. But just as removing the splinter is necessary for the healing to begin, removing algorithm-repetition from the goal of mathematics class will shift the conversation about teaching in a way that will help us refocus on what matters most: helping students learn to think and reason mathematically

FREQUENTLY ASKED QUESTIONS

Q: But, Pam, I learned all of those math-y things *because* I learned the algorithms. When I use an algorithm, I am not mimicking, I'm thinking mathematically. We need to teach those algorithms because that is how I learned the math I know and learned to think the way I do.

A: This is a very tricky conversation. There are subtle things at play here. Would you consider . . . Is it possible that you had natural inclinations to pick up on patterns and relationships, so that when your teacher showed you an algorithm, you created many mental connections with things in and around that algorithm? And since you did, you might now associate all of those self-made connections with learning that algorithm. The mathematical connections like place-value, magnitude, rounding, and friendly numbers might be inextricably linked (in your experience) to the steps of an addition algorithm and the experience of learning it because you were thinking through the steps and making sense of it using your natural talent.

Might you be willing to consider that all of those extra mathematical connections had less to do with being shown steps to repeat and more to do with your natural proclivity to pick up on and use patterns? And if that's true, might it be possible that you could have learned far more, faster, if someone had been actively, purposefully helping you develop those connections? If you did it all on your own, imagine what you could have done with purposeful, expert guidance.

(Continued)

(Continued)

Mathematicians of old and the random nonmathematician ran into the same patterns in life but the mathematician noticed, used, built on them. The random nonmathematician did not. You were able to do what you did, recognize patterns and make connections when presented with an opaque algorithm, without that expert guidance. You know that most students can't do what you did with the same (lack of) support. Many of us could not—we bought into what we were told: memorize and mimic. And even if we tried to make sense of it (pick me), we didn't have the natural talent to do it without those patterns being made explicit.

The question is not whether or not algorithms work as teaching tools for most of the population. They just don't. The question is whether or not we can teach most of the population *and* still give advantaged students like yourself what they got from learning through algorithms.

The answer is that we can, and we can give students like yourself so much more. And the best way to teach it is to high-dose everyone with those patterns.

Q: What if algorithms are required in our standards?

A: Great question! I won't pretend this is trivial, but there are solutions. You can meet your standards and teach real math-ing. This will become clearer throughout the book. Keep reading, and we'll wrap it up in Chapter 7.

BEING TRAPPED BY ALGORITHMS

I want to tell you about a few people I know who were trapped by a math education that did not focus on empowering students to use what they know to figure out what they don't know yet. This misdirection in math education almost always stems from an addiction to algorithms as teaching tools.

At one point in graduate school, I was a teaching assistant for a large college algebra class. The 300+ class met three times a week in a large hall, and students could visit me during my office hours for help with homework and studying for tests. Most of the students who came to me were elementary education majors. They would lament, "I hate this class! Please help me get through it so I'll never have to take math again." They had been trapped by an exclusively speed-and-accuracy-focused math education.

Years later, elementary and middle school education majors in my math methods courses would complete a math autobiography as part of their first assignment. Over 90% of my students would introduce themselves as wanting to teach kindergarten, first grade, or maybe second grade because they did not believe they could teach fractions. They had been trapped into believing they could only teach young grades because of their anxiety around mathematics.

One of my goals every semester is to help my students leave my class ready and excited to teach the grade level they actually want to teach, not just the one they previously thought was the highest math they could teach. Each semester it's one of the best things that happens when students happily, confidently report seeking and obtaining those positions.

Kyle Pearce, cohost of the *Making Math Moments That Matter* podcast and cofounder of the Make Math Moments company, has told the story before that he was a university student when he realized that he didn't really understand mathematics. His mathematics education had trapped him into thinking he was good at math and he was severely disappointed to find that his preparation was so insufficient. His powers of memorization were good, but he hadn't developed the complexity of mathematical thought his professor was looking for. Kyle is now a top-notch coach, math teacher educator, and task designer (Pearce & Orr, 2020).

I myself hit a wall hard in advanced university math courses. I had more than a decade of being rewarded for superb speed and accuracy, only to have my Abstract Algebra professor tell me, "Oh, we don't do that here," when I asked him to give me the proofs ahead of time: "If you'll just tell me what proofs will be on the quizzes and tests, I will memorize them and spit them back out on the assessments perfectly." He just shook his head. I had been trapped by excellent grades for mimicking algorithms into believing I was doing real math when in reality I had been succeeding at rote-memorizing and mimicking, not math-ing at all. I had not developed the sophistication of thinking I needed and wanted.

These are just a few examples of people who have been trapped by algorithm-focused math instruction. We'd memorized piles of algorithms, but instead of climbing higher with each one, we were drowning in more and more disconnected facts and procedures. We'd learned, as the Dutch mathematician Hans Freudenthal said, that school mathematics is like the "fossilized remains" of real mathematics (Freudenthal, 1973). In being taught this way, we'd gained, if anything, the perverse ability to get answers without any of the foundation required to make use of them.

WHAT IS AN ALGORITHM?

So let's make sure we're on the same page with vocabulary. An *algorithm* is defined as a series of steps to solve any problem of a particular kind. It is the same method for every problem, regardless of the numbers or structure. All the steps, every time (Carpenter et al., 1998; Kamii & Dominick, 1998; Plunkett, 1979; wolframalpha.com, 2024).

This definition of an algorithm is almost universal in science, statistics, computer science. Over the past few years in mathematics education, some people have started to use the word algorithm *when* strategy *would be more precise. This muddies the water. See page 23 for more on the distinction between algorithms and strategies.*

Algorithms are a general solution, which means an algorithm can solve even the gnarliest of problems. Algorithms are often opaque, where the place-value, magnitudes, and meaning are hidden behind the scenes, cleverly embedded so the user does not have to deal with the complexities that are occurring. This is the beauty, cleverness, and remarkableness of the algorithms. It is also what makes them terrible teaching tools.

The algorithms I refer to as the traditional algorithms and use as examples in Chapters 3–6 all share the commonality that they can be counted on to reduce any problem of a type to single-digit arithmetic. For example, the traditional North American multiplication algorithm will turn any multiplication problem, no matter how many digits are involved, into a series of single-digit multiplication and single-digit addition problems (see Chapter 4). This is very powerful, but also inherently limiting.

Memorizing one algorithm is usually no help with memorizing the next one. Knowing the steps of a traditional multiplication algorithm will not help with memorizing the steps of the traditional long division algorithm.

This means that for many students math class becomes increasingly frustrating and difficult to manage. As they progress through the grades, what they learned last year does not help them understand what they are learning this year.

Because students do not actually have to understand what is going on to perform the steps, students can use less complex reasoning than those problems could help develop. "Fantastic!" critics cry. "Students will be able to do so much math without really having to deal with the complexities (i.e., learn anything).

This is a desired outcome! More students doing more math. Who wouldn't be in favor of that?"

But students using algorithms are only *mimicking* more math. As Liljedahl (2021) observed in his book *Building Thinking Classrooms*, "Everywhere I went I saw the same thing—students not thinking and teachers planning their teaching on the assumption that students either couldn't or wouldn't think" (p. 12).

Our goal in teaching mathematics cannot be to make things *easy*, particularly if it means sacrificing long-term growth for short-term answers on homework and test scores. If that was acceptable, we could hand out the answer key and call it a day. We as teachers know this instinctively. Some amount of struggle is necessary for learning. But that does not mean all struggle is useful or created equal. Struggling to memorize and mimic is effort spent *now* for more confusion *later*. Grappling with and making sense of real math pays huge future dividends. Real math is knowledge that builds on itself.

TRY IT

Think about the algorithms that you teach your students. Having that list in the forefront of your mind will be useful as you continue to read.

Our world needs thinkers and reasoners, so our world needs a math class that trains thinkers and reasoners. Liljedahl (2021), who encourages us to *Build Thinking Classrooms*, said, "My goal from the outset was to get students to think. . . . Thinking is a necessary precursor to learning, and if students are not thinking they are not learning" (p. 296).

That is why I object to using algorithms as teaching tools. As Hurst and Huntley (2018) found, "most students in [their] sample are 'prisoners to procedures and processes' irrespective of whether or not they understand the mathematics behind the algorithms."

Algorithms are amazingly powerful mathematical inventions that are essential to much of modern technology. The issue is that the skill of developing and finding uses for algorithms is vastly different from the skill required to follow the steps of an individual algorithm. The former can program a computer to land a rocket on the moon; the latter can be replaced by that 50-year-old computer landing that rocket on the moon.

The ability of a human mind to create an algorithm is amazing; the ability of a computer to execute an algorithm is revolutionary; and the damage done by using algorithms to teach mathematics is incalculable.

FREQUENTLY ASKED QUESTIONS

Q: But mathematicians use algorithms, right? So if we want our students doing math like mathematicians, our students should be using algorithms, right?

A: In a study where mathematicians were given problems to solve, they used an algorithm only 4% of the time. That means that 96% of the time mathematicians reasoned through the problems, using relationships they know. Mathematicians create algorithms, study algorithms, compare algorithms. They don't use them to compute (Dowker, 1992).

Q: What if my standards require algorithms?

A: The short answer is that in most circumstances you can meet your standards and avoid the trap of algorithms. The longer answer is in Chapter 7 and will make more sense after reading Chapters 2–6.

Q: Why do you refer to the *standard* algorithms as *traditional* algorithms?

A: There is nothing standard about the algorithms that have become traditional. The word *standard* denotes "one and only" and gives too much weight and credibility. People are often shocked to find there are several different algorithms commonly in use around the world. My mother grew up in Switzerland and does division completely differently than what I learned. The subtraction algorithm my eldest son independently created in second grade is the same as the method taught in many Latin and South American countries.

CONFUSING LOGICAL-MATHEMATICAL KNOWLEDGE FOR SOCIAL KNOWLEDGE

By now you are probably wondering why teaching algorithms evolved as *the* way to solve problems in math class. To answer this question, let's parse out the difference between social/conventional knowledge and logical-mathematical knowledge.

Child development psychologist Jean Piaget suggested there are three types of knowledge (Piaget, 1974):

1. **Physical knowledge:** The understanding of the physical world.
2. **Logical-mathematical knowledge:** The understanding of being able to solve problems and perform analytical reasoning.
3. **Social knowledge:** The understanding of societal norms and conventions.

The last two types are important when we discuss learning math: logical-mathematical knowledge and social knowledge.

The trouble in math education is that we have a history of treating mathematics as all social knowledge. In reality, most of mathematics is logical knowledge. Next, let's make sense of the difference and how they both are best taught.

Historically, mathematics was only for wealthy individuals who could afford expensive educations. Throughout history, math-y people had developed mental relationships that culminated in really cool general algorithms. As education democratized, schools tried to bring math to everyone. Schools did so with textbooks that lifted those algorithms from that body of work and handed them to teachers and students as if the algorithms are all the math there is to learn and not one small application of it. The resulting misalignment of scope resulted in the limiting idea that answers are the best evidence of learning math. That has been passed down, and over time it became the picture of what math is. Even today, as people write scope and sequences, lessons, and textbooks, they tend to cut up the interconnected web of relationships into tiny memorizable pieces, making mathematical ideas into falsely linear sequences, giving all things equal weight, and making it all about answer-getting. That put us very far away from what math-ing actually is.

Before, we just didn't know how to pull back the curtains and help students develop mathematical reasoning. Now, we do. To learn how, read on!

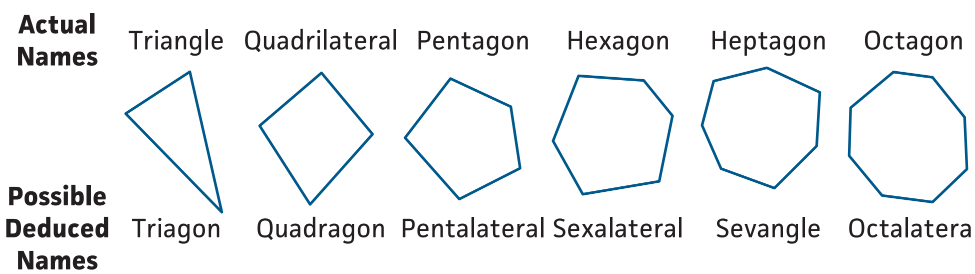
SOCIAL KNOWLEDGE

Social knowledge is that which we deem to be so.

Another way to say this is that someone suggested social knowledge as knowledge, and over time *by convention we all adhere to that idea*. In math, there is a small set of things that must be told—they cannot be figured out. This set consists mostly of vocabulary and notation that history has tapped as *the* way to say it or write it. Given that these conventions are not figure-out-able, students would only be able to guess, and more often than not they would guess incorrectly.

For example, we cannot ask students to reason to find the name of a four-sided polygon (many angled figure). If they use patterns, 10-sided is decagon, six-sided is hexagon, five-sided is pentagon, so surely a four-sided polygon is called a quadagon? A fouragon? Squareagon? Oh, actually we have the tradition of calling them quadrilaterals (four-sided figures). We can't reason our way through tradition. Good grief! What is a three-sided polygon called? A tri-ilateral? A thrice-agon? (See Figure 1.1.)

FIGURE 1.1 ● Social Knowledge Versus Possible Figure-Out-Able Names for Common Shapes



TIP

Teach vocabulary just in time, not just in case. When you teach vocabulary just in case, students learn lists of terms and definitions beforehand, in case they will need them. Rather, teach vocabulary just in time. Give students experiences where they are begging for a way to describe what they are thinking about and dealing with. Teaching just in time means hanging terms on already constructed logical-mathematical knowledge.

Notation is another example of social knowledge. For example, parentheses mean multiplication, $3(4) = 12$, except when they don't:

- $f(x)$,
- $f^{-1}(x)$
- the point $(2, 3)$
- interval notation $(-\infty, -2)$

We can't figure out what the notation is supposed to mean without some social help. Students could guess, but they'd likely guess incorrectly because there is no way to logically reason or use what they know.

Let's not make students guess about the parts of mathematics that are social knowledge. This set of knowledge is the things we should purposefully tell students with intentionality and straightforward clarity.

LOGICAL-MATHEMATICAL KNOWLEDGE

Most of mathematics, however, is *logical-mathematical knowledge*—that which must be experienced, connected, reasoned, *figured out* in order to actually learn, own, and use meaningfully. To reason mathematically, these things cannot simply be told, memorized, or mimicked. The best way to teach students mathematics is to teach them to mathematize, to do the kinds of thinking involved in math-ing.

This logical-mathematical knowledge is the power to see 99×675 and, rather than funneling into an algorithm of steps to parrot, instead use Multiplicative Reasoning to recognize that 99 groups is one less than 100 groups, and therefore the answer is one less 675 than 67500.

Such connections are everywhere in mathematics, even in areas you might think right now can't possibly be figured out. We know *this bit of math* because someone figured it out in the first place, then created the algorithm we might think is the only way to solve it. We know this because such connections define what mathematics *is*.

Are there parts of 99×675 that are social knowledge? Yes, the look of the numerals and the multiplication symbol are social knowledge, and that is a fruitful teacher discussion—the parts of math that must be told and the parts that must be experienced and worked through with logic. I invite you to consider that the set of things that are social knowledge in mathematics is *far smaller* than we have been led to believe.

This conversation is in part difficult because so many of us were taught math as *all social knowledge*: Wait to be told what to do, then rote-memorize and mimic *everything*.

If we are teaching a part of mathematics with a mnemonic, rhyme, or story, we are treating it as social knowledge—that which must be told and memorized.

UNPACKING THE CONFUSION

Learning the names of rivers that traverse a country or all 50 capital cities in the United States is social knowledge. These things must be rote-memorized.

What about multiplication facts? Is something like 7×4 social knowledge? Pause here. Take inventory of what you think. Are multiplication facts something to be clearly told and then rote-memorized (social)? Or are they to be connected and reasoned about (logical-mathematical)?

Multiplication facts are not like the random names of polygons, rivers, or capital cities. They are internally consistent and logically built from each other. Eight groups of eight, 8×8, is 64, and one more group of eight, 9×8, is 72, and $64 + 8 = 72$. But a student might never realize those connections if they are instructed to rote-memorize each fact. Imagine what a student thinks about the facts if they "learn" the facts as is suggested in a "learning" video program I encountered on YouTube. I've summarized a part of it for you as follows:

> Learn 7×4 in under 10 minutes! All you have to do is memorize this story. Mrs. Week (7) sits on a chair (4). She goes fishing and catches 2 boots and 8 fish. You got it: $7 \times 4 = 28$.

The video even instructs students to not remember the story wrong—she didn't catch 8 fish and 2 boots. Suggesting that 2 boots and 8 fish is not the same as 8 fish and 2 boots sends the message that this is all arbitrary and that addition is not commutative. It also sends the message that 28 is made up of 2 and 8, not 20 and 8. This is a prime example of sacrificing your future multiplicative reasoning self for a current third-grade story repeater. Sure, some kids will rote-memorize facts more easily with a story. But then that's all we get—students repeating nonsensical stories pretending they are doing math. Consider the chance students have to *understand* fractions if they believe multiplication facts are arbitrary, disconnected, random *vocabulary*. They are trapped.

To help explain the prevalent obsession with rote-memorizing multiplication facts, consider this. When many of us were learners in the midst of performing the multiplication algorithm over and over again, it became painfully clear that crunching through each of those problems was easier if you had the single-digit facts at your fingertips. This striking and strong memory leads us to think, "To help students, let's make sure that they rote-memorize all of those facts." In our embodied memory of our student experience, we didn't need to understand more, so an alternative was not even on our radar.

Of course, we want students to know their multiplication facts! But the mathematical reality is that students need to *more* than know them. Students need to travel the mental path of figuring the facts often so that those paths become well traveled. Once we can get students reasoning using the connections between multiplication facts, they learn those facts *and* the ones those facts connect to *and* they develop Multiplicative Reasoning at the same time. "Providing students with opportunities to think about things differently, find similarities and differences, and evaluate which approach is best, all enhance the brain's construction of new learning" (Jensen & McConchie, 2020, p. 175). Multiplicative Reasoning is the goal. As we develop Multiplicative Reasoning, owning the facts becomes a natural by-product.

Multiplication facts are one of the first opportunities to help students develop Multiplicative Reasoning. Multiplicative Reasoning looks like finding 9×8 by thinking about $10 \times 8 = 80$, but since that's too much, removing one 8, for 72. This helps with thinking about things like 49×6 by thinking about $50 \times 6 = 300$, but since that's too much, removing one 6, for 294. "This kind of deep dive into the learning ensures the brain understands the wider or deeper context and not just isolated or rote facts" (Jensen & McConchie, 2020, p. 175). This reasoning Jensen and McConchie describe is logical-mathematical and essential for everything that comes afterward, like fractions, proportions, functions, etc. Chapter 2 will continue to illuminate how to develop reasoning.

The issue with memorizing multiplication facts the same way you would memorize capital cities in a geography class is that *mathematics class is not a geography class*. Rivers, mountain ranges, capital cities' names are social knowledge. To know these is to memorize. The multiplicative connections and relationships between facts are logical-mathematical knowledge. To know these is to *math*.

HOW TO TEACH

If everything in mathematics were social, then we would clearly need to tell students all of it.

This just isn't the case.

Most mathematics is not social knowledge. And because most mathematics is not social knowledge, it is unhelpful to give procedures and hope students guess at the reasoning underlying them when it is the reasoning that is important. Students will get answers, but they probably won't build reasoning.

We need to approach the teaching of mathematics with the recognition that most of the material is logical, adheres to patterns, and can be deduced from other knowledge that is already understood. This does not mean that students should do all of the connecting on their own—I am certainly not an advocate for unfocused, anything goes, fumbling, fuzzy math teaching.

While these essential, extremely powerful connections are everywhere in mathematics, that does not mean they are so obvious that students should be left to their own devices to discover them. I'm not advocating that we turn classrooms into directionless vacuums, where students are left to their own devices to forage for math.

Students benefit from a "more knowledgeable other" (Vygotsky, 1978) in a classroom where teachers are the guides who craft experiences, give feedback, mentor, and support every student mathematician. Students need teachers who assess what fledgling knowledge students already have and help students build on that prior understanding. Not as a minor item on a checklist before a lesson but as the foundation on which the entire classroom experience is built. They need teachers who know the mathematical relationships and how to help students develop these relationships and connections for themselves.

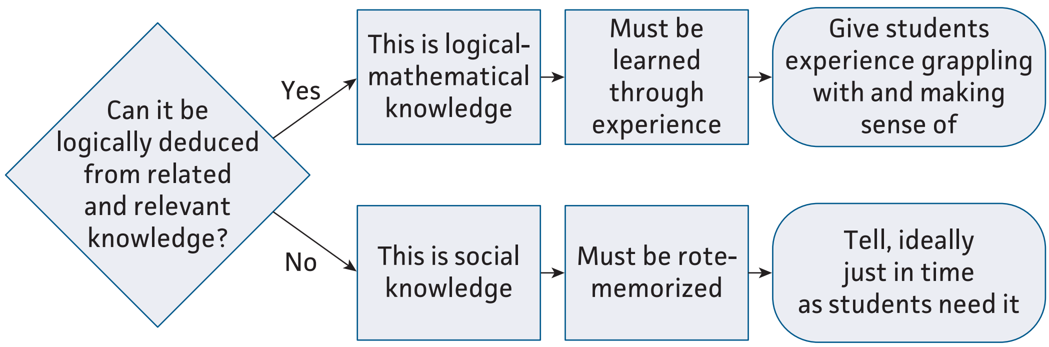
Kim Montague, my pithy cohost on the Math Is Figure-Out-Able *podcast, gives us the challenge to "Know your content, know your kids." (Montague, 2021) This is not just a side-note, catchy phrase. Knowing your content and knowing your students are the bedrocks on which we build the foundation of our math classes. Most of this book is to help you build your content. Equally important are knowing your kids, assessing what they know, how they learn, what motivates them, their culture, and their interests. Giving your students open enough tasks so they all have access, and they are all challenged in appropriate ways, is the exciting and important work of building on the content with which they come to you.*

Efforts at discovery-based teaching frequently end up with kids doing the best they can alone trying to guess what's in the teacher's head. That can feel callous, and it is not helpful. Often that triggers a reaction of frustration from parents and colleagues: *Why hold back on students? Just tell them the things. Don't make them reinvent math—just tell them what to do in the clearest possible way. Give them plenty of practice to make that mimicking stick.* As teachers we might be lulled into a false sense of security, believing that if students are getting answers, and parents are happy, everything is fine. We might also believe that all math is social knowledge. Therefore: tell, memorize, mimic, practice mimicking.

Except, as we've just discussed, everyone is not fine when students are rote-memorizing and mimicking, fake math-ing. Even the "successful" students quickly find that the "math" they've learned is nearly useless outside a fake math classroom. Students' mimicking muscles are getting a workout, but students' mathematical reasoning abilities are staying stagnant or atrophying. Many of us—teachers and parents—are unknowingly perpetuating a fake math myth.

If we clearly understand the parts of mathematics that are social versus those that are logical-mathematical, deciding how to teach each part is simple: Construct most of mathematics by mathematizing with students, giving them experiences so their brains can make the connections. Tell only the small set that is social knowledge (see Figure 1.2).

FIGURE 1.2 ● A Decision Tree to Inform Teaching Practices



It is real work to learn to differentiate between what is social and what is logical, between the arbitrary words and notation we've chosen and the real depth and expanse of mathematics. Between what is window-dressing we only need for communicating with each other and what is figure-out-able.

But that's why you are reading this book, right?

WHAT IS MATHEMATIZING, MATH-ING?

What does it look like and feel like to mathematize, to math the way a mathematician *maths*?

Fosnot and Dolk (2001a, 2001b, 2002, 2010) in their Young Mathematicians at Work *series have a chapter titled "Mathematics or Mathematizing?" in which they discuss the purpose of mathematics education. And Crayton (2026) is one of those who has made math into a verb: math-ing. Freudenthal suggested, "What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (Freudenthal, 1968, p. 7).*

Here's a glimpse into mathematizing: What is 99 plus anything? What is 99 times anything?

TIP

When you see problems like these in the book, solve them before you read on. Getting a sense of the relationships involved will help you make sense of the commentary that follows.

Think about the problem 99 + 47. How do you reason about that sum?

- Could you think about 100 + 47? Add a bit too much, so adjust one back?
- Could you think about *one more* plus *one less*, 100 + 46?

We can use both of those strategies to reason about 99 plus *anything*.

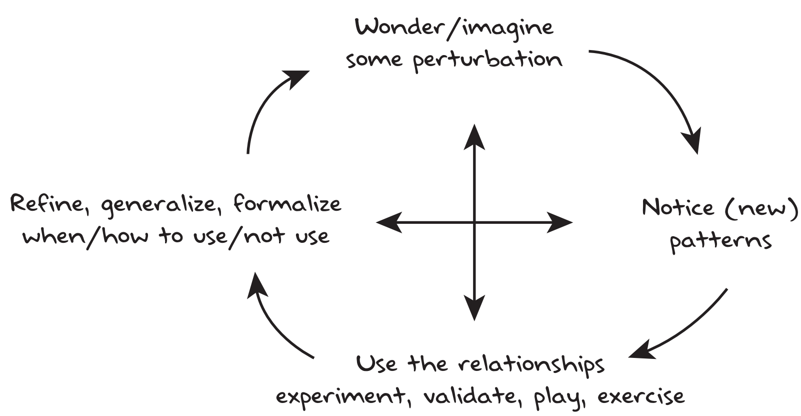
Think about the problem 99 × 47. How do you reason about that product?

- Could you think about 100 × 47? That's too much, so adjust one group back?

We can use this *over* strategy to reason about 99 times *anything*.

These are examples of using what you know, adding 100 or multiplying by 100, to reason through a problem. This is the work of mathematizing, of math-ing. This kind of work strengthens your brain and builds your capacity to deal with more complex ideas and relationships (see Figure 1.3).

FIGURE 1.3 ● The Work of Math-ing



Mathematics teaching at its best means to give students something mathematically important to think about, where they can wonder and imagine. This leads to helping students notice patterns that students can use to experiment, validate, play, and exercise their fledgling ideas. With expert teaching help, they can then refine, generalize, and formalize their thinking about when and how to use or not use the relationships, which leads to the opportunity to wonder about new things on their horizon.

We can mentor students to mathematize by helping them develop such mathematical strategies.

FREQUENTLY ASKED QUESTIONS

Q: You just used the easy number 99. Of course, you can reason with such easy numbers. What about all the other numbers? Don't we need algorithms for those numbers?

A: No. For any number or problem that is so complicated that you might need to reach for an algorithm, you could instead reach for a calculator. The major mathematical strategies that I outline in this book give students access to *enough* problems. That's why these strategies were chosen: to build mathematical reasoning and to give students power over *enough* problems. For more discussion on problems that are worth solving without a calculator, see Chapter 7.

WHAT IS A STRATEGY?

We just discussed a strategy of adding or multiplying by 99. But what is a *strategy?* Using a strategy means letting the numbers or structure in a problem influence how you solve it. Your strategy is how you use the relationships you already know to solve a problem. It means you don't use the same method for every problem because the numbers are not the same, and that you use only what you need to solve the problem, not someone's prescribed steps (Fosnot & Dolk, 2001a; Fosnot & Dolk, 2001b; Wright et al., 2006). Where algorithms are often opaque (difficult to see what's happening behind the scenes), strategies are transparent because you are using relationships *you know* to reason through the problem.

Mathematical strategies are approaches to problem solving that are distinguished from each other by the underlying understanding of mathematics required to use them. These strategies are the ways that naturally math-y people use the patterns they intuitively notice. Just because the rest of us did not pick up and use the patterns on our own, does not mean that we cannot. We can all benefit from deliberate, higher doses of the patterns.

Unlike algorithms, learning strategies are synergistic. Where each new algorithm to memorize is another series of steps to potentially misremember and confuse with each other—was it keep, change, flip, or keep, change, change?—strategies are mutually reinforcing. The more relationships you own, the more strategies you learn, which in turn builds more relationships. This positive feedback cycle is what learning real math-ing is all about.

For example, there are four major subtraction strategies that represent the mathematical relationships a student needs to own. To be clear, I'm not advocating learning four different algorithms instead of one. Decades of classroom experience have shown that building strategies, by leveraging and developing student intuition instead of destroying it, takes far less effort individually and pays far more dividends than even one algorithm required to solve the same class of problems.

Unlike traditional algorithms, major strategies do not reduce problems to single-digit arithmetic. While this sounds like a weakness, it is actually a massive learning advantage. It enables strategies to take advantage of what students already know beyond the most basic of single-digit operations. Instead of reducing 120×9 to a long series of single-digit problems, we can leverage what we've learned in math class since third grade to instead solve $120 \times 10 = 1200$, and $1200 - 120 = 1080$. The former stops building brains the moment single-digit operations are conquered, while the latter lays the foundations for proficiency of fractions and Proportional Reasoning.

FREQUENTLY ASKED QUESTIONS

Q: But traditional algorithms are so efficient and save time, right?

A: If the argument is that algorithms save time because a student uses one way to solve a problem every time, then we're not considering the time spent teaching and practicing these algorithms. We're also not considering the time lost in a year remediating students who are unable to mimic these procedures, the years lost in a student's mathematical journey because they quit math, misunderstanding what mathematics really is. By contrast, strategies are natural outcomes of relationships students *need to own*. They are extensions of the mathematical properties that are at the heart of mathematics. Strategies offer students choice so they are able to consider which relationships they want to work with rather than waste time attempting to remember an algorithm they never understood. Given any problem that's reasonable to solve without a calculator, we can be as efficient as a traditional algorithm or, most of the time, *more* efficient. Many examples of this efficiency of strategies follow in the rest of the book.

Q: It sounds like you're not advocating for direct instruction or inquiry. What are you advocating for?

A: I'm advocating for a shift in goals—from mimicking algorithms to developing mathematical reasoning (which includes content). With that new goal in mind, teaching then becomes: good guided inquiry for everything that is logical and clearly telling for the bit that is social. Teachers have clear goals: help students grapple *long enough*; guide students to important generalizations through purposefully crafted discussions; anchor learning; and keep building on that learning to move the mathematics forward using open access enough tasks that all students continue to have access and continue to be challenged. By doing this, students are not just solving problems correctly and efficiently, but also more sophisticatedly. This allows students to be more successful longer. More on this in Chapter 2.

For any problem that's reasonable to solve without a calculator we can be as efficient as a traditional algorithm or, most of the time, more efficient.

Conclusion

The purpose of math class is to develop mathematical reasoning, not mathematical answer-getting. What we need are not mere calculators but thinkers, do-ers of mathematics to solve problems we have yet to encounter. Our role as teachers is to guide and support students as they develop their mathematical reasoning, not rotely mimic algorithms that only provide answers to existing problems. Knowing the traps of algorithms empowers us to make other choices.

Discussion Questions

1. What is the difference between an algorithm and a strategy? The book will further differentiate these, but what are your current thoughts?
2. What is the difference between logical-mathematical knowledge and social knowledge?
3. What is an example of a bit of mathematics that is social knowledge and therefore told to students? Do you and your colleagues agree on this?
4. How were you taught the multiplication facts, as logical-mathematical or social knowledge? How did that affect your perception of the nature of mathematics?
5. What do you think of *math* as a verb? What does it mean to you *to math*?

TRY IT IN YOUR CLASSROOM

99 Plus Anything

Purpose

The purpose of this short interview is to become more aware of the strategies many people use intuitively. It also gives you the opportunity to practice your questioning and listening skills, and your parsing of people's mathematical thinking. Seek to pull out people's reasoning, teasing out what they mean. Ask, don't tell.

This can be quite challenging if:

- you're used to listening solely for correct answers or correctly mimicking steps
- you're like I was, with the algorithm as the sole method you use to solve problems
- you have yet to try to figure out other people's alternative strategies

Use these interviews to open your horizons in a low-stakes environment. Just have fun!

Routine

- Interview several people (your family, neighbors, community members, students, colleagues, anyone willing).
- Ask, "What is 99 plus anything?"
- If the person is confused, clarify, "What is 99 plus any number?"
- If more clarification is needed, add, "Pick an ugly number." (Smile when they choose a number that ends in 7) and ask, "What's 99 plus your [37]?"
- Listen, watch, ask to hear what's happening in their head.
- Try to repeat back to them what they did, putting your own words to their strategy.
- Try their strategy with a different number and ask them if you understood their thinking.

Important to Consider

Make this a casual conversation, not an interrogation. You don't want students, friends, and family members to feel like they're on the spot, especially if they are in front of their peers. Reassure them by suggesting that you're practicing learning how people think about math-y things when they're not necessarily trying to please a math teacher, the way they would actually reason. Ask clarifying questions until you understand what they are thinking. If they start to tell you about an algorithm, don't make them explain those steps. Instead, gently probe for what they might do without those steps.

Some people may add the tens, add the ones, then add those sums. Others may use an over strategy, finding $37 + 100$ and adjusting back 1. Some people may give and take, taking 1 from the 37 to give to the 99, making an equivalent problem, $36 + 100$.

Extension

Depending on the age or experience of your interviewee, you could ask any of the following:

- 9 plus anything
- Anything minus 9
- Anything minus 99
- 9 times anything
- 99 times anything
- Adding $9\frac{9}{10}$ to anything