WHAT YOUR COLLEAGUES ARE SAYING . . .

This most recent addition to the *Figuring Out Fluency* series is outstanding! It provides guidance and support for building foundational fluency skills and practical ways to implement meaningful assessment and joyful practice. This book could shift mindsets about what it truly means to be fluent in math.

Deborah Peart
CEO & Queen Mather of My Mathematical Mind
Ocala, FL

SanGiovanni, Bay-Williams, and Katt have done it—the ultimate fluency resource! The focus is on helping students through their struggles using the lens of the foundational knowledge and skills needed for computational fluency. This is a perfect blend of lesson seeds, routines, games, and centers that provide first instruction or intervention for every student. This book, coupled with self-reflection, humility, and productive struggle, provides a rock-solid foundation so all students can weather any storm!

Ron Perry
K–4 Math Specialist, Heim Elementary, Williamsville Central School District
Williamsville, NY

Figuring Out Fluency—Ten Foundations for Reasoning Strategies With Whole Numbers is a game changer! In this companion to the *Figuring Out Fluency* series, the authors provide teachers with hands-on, engaging resources to support teaching and learning of foundational fluency concepts. This resource uses games, routines, and centers to help focus on reasoning about fluency concepts!

Latrenda Knighten
Mathematics Curriculum Supervisor, East Baton Rouge Parish School System
Baton Rouge, LA

Undoubtedly the most impactful book in the series, *Figuring Out Fluency—Ten Foundations for Reasoning Strategies With Whole Numbers* is a must-have for every elementary teacher. It explains not only the foundational skills and concepts needed for fluency but also the how and why they are necessary for strategic thinking. The look-fors and practical, easy-to-implement ideas for instruction and practice will help students build their mathematical identities and grow as mathematical thinkers.

Brenda Dzwil & Holley Duffy
Instructional Coaches, Newington Public Schools
Newington, CT

Get ready for a comprehensive yet accessible read for teachers to lay the roadmap for equipping students with indispensable foundational skills for learning fluency strategies. The authors outline ten foundations and supplement them with descriptions, examples, and teacher-friendly language to implement in the classroom tomorrow. The variety of included activities will create a book filled with dog-eared pages that teachers will reach for regularly.

Marci Ostmeyer
Professional Development Director, Educational Service Unit 7
Columbus, NE

The ten foundations for reasoning strategies shared in this book are practical and designed for immediate classroom application. They encompass methods to teach, practice, and assess fluency, which foster a supportive learning environment that cultivates a positive math identity among students.

Janel Marr
Math/STEM Resource Teacher, Windward District
Kānéohe, HI

This book provides teachers with an understanding of how to support and solidify foundational skills for students. It gives teachers a toolkit of assessments, games, and centers that are easily accessible and effortless to implement. It also challenges the traditional classroom, pushing the teacher to become more of the facilitator while encouraging an equitable education for all students.

Marissa Giangrosso
Instructional Coach
Blue Springs, MO

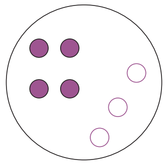
I own every book in the *Figuring Out Fluency* series, and this is the missing piece. This book examines the why and how number sense works in building procedural fluency with tools both for student practice and assessment of understanding. The structures and strategies are explained in a way that new and veteran teachers can implement in the classroom.

Christina Worley
K–5 Math Curriculum Developer, St Lucie Public Schools
Port Saint Lucie, FL

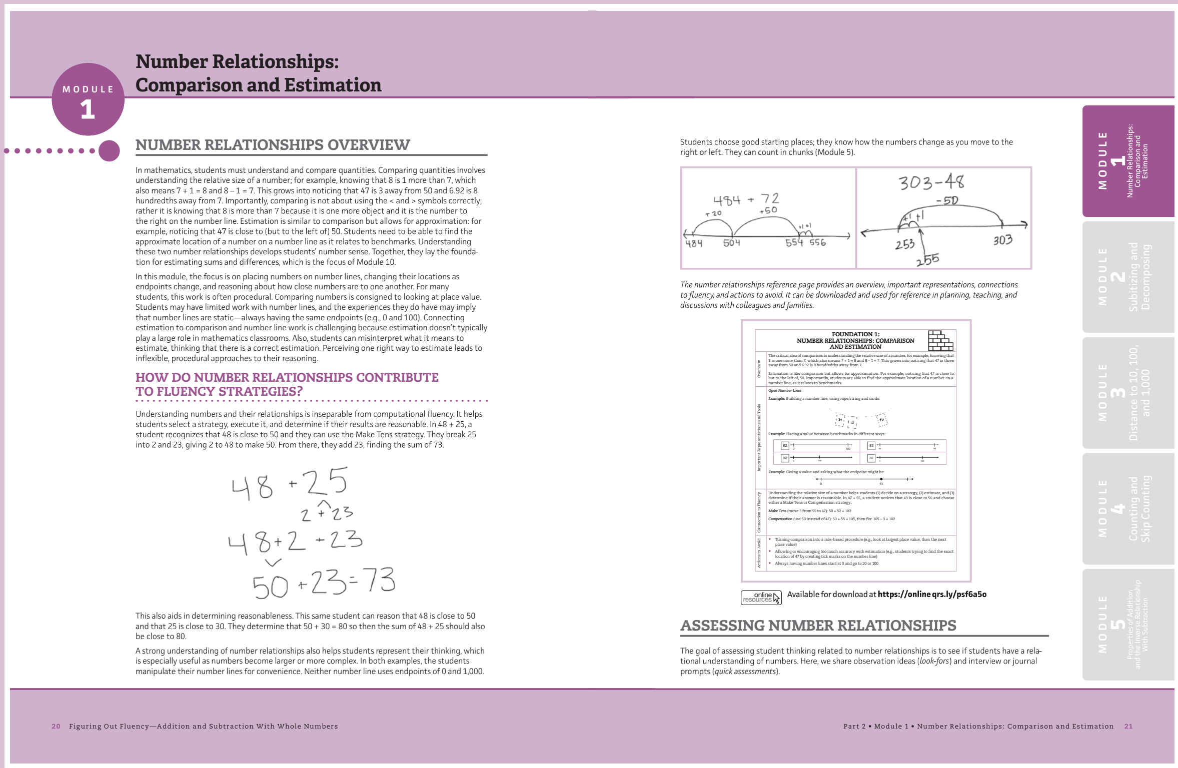
Figuring Out Fluency—Ten Foundations for Reasoning Strategies With Whole Numbers at a Glance

Building off of *Figuring Out Fluency*, this classroom companion explores the foundational ideas that are essential to numerical reasoning and number sense.

FIGURE 7 • Foundations for Computational Fluency

FOUNDATIONS MODULES	WHAT IT IS (IN BRIEF)	EXAMPLE PROMPT
1. Number Relationships: Comparison and Estimation	Knowing approximately where a number is in relation to other numbers	Please place 28 on a number line. Use various endpoints [0, 50], [20, 30], or [0, 100].
2. Subitizing and Decomposing	Subitizing is visually seeing subsets of the whole to determine the whole; decomposing is starting with the whole and determining the parts	 How many dots? How do you see them? Decomposing: There are 10 turtles on two logs. How many might be on each log?
3. Distance to 10, 100, and 1,000	Seeing how far a number is from a benchmark (over or under)	How far is . . . • 8 from 10? • 37 from 40? • 107 from 100? • 288 from 300?

Overviews of each foundation show how it contributes to fluency strategies, how to assess it, and how to teach it explicitly either as first instruction or intervention.



MODULE 1

Number Relationships: Comparison and Estimation

NUMBER RELATIONSHIPS OVERVIEW

In mathematics, students must understand and compare quantities. Comparing quantities involves understanding the relative size of a number; for example, knowing that 8 is 1 more than 7, which also means 7 + 1 = 8 and 8 – 1 = 7. This grows into noticing that 47 is 3 away from 50 and 6.92 is 8 hundredths away from 7. Importantly, comparing is not about using the < and > symbols correctly; rather it is knowing that 8 is more than 7 because it is one more object and it is the number to the right on the number line. Estimation is similar to comparison but allows for approximation: for example, noticing that 47 is close to (but to the left of) 50. Students need to be able to find the approximate location of a number on a number line as it relates to benchmarks. Understanding these two number relationships develops students' number sense. Together, they lay the foundation for estimating sums and differences, which is the focus of Module 10.

In this module, the focus is on placing numbers on number lines, changing their locations as endpoints change, and reasoning about how close numbers are to one another. For many students, this work is often procedural. Comparing numbers is consigned to looking at place value. Students may have limited work with number lines, and the experiences they do have may imply that number lines are static—always having the same endpoints (e.g., 0 and 100). Connecting estimation to comparison and number line work is challenging because estimation doesn't typically play a large role in mathematics classrooms. Also, students can misinterpret what it means to estimate, thinking that there is a correct estimation. Perceiving one right way to estimate leads to inflexible, procedural approaches to their reasoning.

HOW DO NUMBER RELATIONSHIPS CONTRIBUTE TO FLUENCY STRATEGIES?

Understanding numbers and their relationships is inseparable from computational fluency. It helps students select a strategy, execute it, and determine if their results are reasonable. In 48 + 25, a student recognizes that 48 is close to 50 and they can use the Make Tens strategy. They break 25 into 2 and 23, giving 2 to 48 to make 50. From there, they add 23, finding the sum of 73.

This also aids in determining reasonableness. This same student can reason that 48 is close to 50 and that 25 is close to 30. They determine that 50 + 30 = 80 so then the sum of 48 + 25 should also be close to 80.

A strong understanding of number relationships also helps students represent their thinking, which is especially useful as numbers become larger or more complex. In both examples, the students manipulate their number lines for convenience. Neither number line uses endpoints of 0 and 1,000.

Students choose good starting places; they know how the numbers change as you move to the right or left. They can count in chunks (Module 5).

The number relationships reference page provides an overview, important representations, connections to fluency, and actions to avoid. It can be downloaded and used for reference in planning, teaching, and discussions with colleagues and families.

online resources Available for download at **https://online qrs.ly/psf6a5o**

ASSESSING NUMBER RELATIONSHIPS

The goal of assessing student thinking related to number relationships is to see if students have a relational understanding of numbers. Here, we share observation ideas (*look-fors*) and interview or journal prompts (*quick assessments*).

20 Figuring Out Fluency—Addition and Subtraction With Whole Numbers

Part 2 • Module 1 • Number Relationships: Comparison and Estimation 21

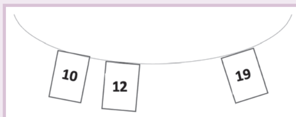
Each foundation module includes teaching activities that help you explicitly teach the strategy.

ACTIVITY 1.2
NEAR AND FAR

Estimating helps students think about the *relative position* of numbers—how close or how far numbers are from one another. This allows students to think about numbers that are close to one another and those that aren't. By doing so, students will develop a better sense of what might be a good estimate: 50 is a better estimate for 45 than 80.

For this activity, gather a string and index cards. The string can be stretched out between two chairs by securing the ends or it can be attached to a wall or board. This string will serve as a number line where students will place number cards. Prepare number cards by folding each card in half and writing a number on each side (you may have heard this referred to as "clothesline math"). The numbers can be within any range that is appropriate for the students.

Place one card's fold over the string, so it hangs and shows the number. Start with a decade number—one ending in zero. Hold up another card and have students determine if it would be near or far from the first number. They will place it on the number line accordingly and provide their thinking about why they placed the card where they did. It might be easier for students to begin with numbers that are relatively close to each other. This allows students to visualize a number line. Shown in the following example, a 10 is placed on the string. Then students determine that 19 is far from 10, so this card is placed on the other end of the string. The next card is shown, a 12, and students determine that 12 is near 10. It is probably best to only place three or four cards at a time on the line. Otherwise, it might become too crowded with cards, which would make it difficult for students to think about the position relative to the initial number.



Continue to place different numbers on the number line and ask students, "Is this number near or far from the number that is on the number line?" Vary the position of where you hang the first number cards on the line, as this will encourage students to visualize a number line and think about the distance between numbers. It is important to note, however, that the exact spacing between the cards doesn't really matter for this activity. Some students may worry about the spacing and try to get it accurate, but that isn't the goal of this activity. Rather, it is to help them think about how close together or how far apart numbers are.

As a follow-up to a whole-class activity or station activity later, students can replicate this activity with partners. Students use a whiteboard to draw a line similar to the physical one used previously. One student places a number along the line and then gives the other student a number to position. The student who places the number says, "___ is near/far from ___." So, if a student writes 40 on the number line and offers the number 87, the second student places 87 further to the right of 40 and says, "87 is far from 40."

ACTIVITY 4.6

Name: "The Count" **Type:** Routine

About the Routine: This quick routine can give you insight into students' proficiency with counting while providing a good opportunity for practice. The Count is a routine that has students estimate and skip count. It is a good opportunity for practicing skip counting by a variety of intervals, which is essential for using the Count On and Count Back strategies. You can extend the routine by ending with a problem connected to the counts. For example, you can have students start with 43 and count on by ones. Then, have them start with 43 and count on by tens. Pose a problem, such as 43 + 48. Have students turn and talk with their classmates about how they can count on by tens and ones to find the sum.

TEACHING TAKEAWAY

You can increase engagement with routines by turning them into challenges or games. For The Count, you can have students write down the number they think they will say or the number they

Routines, Games, and Centers for each foundation offer extensive opportunity for student practice.

ACTIVITY 4.10

Name: 300 Is Perfect **Type:** Game

About the Game: This *300 Is Perfect* game blends estimating and counting. The goal of the game is simple: Players try to get as close to 300 as possible without going over. Each player starts at 0 and rolls a digit. That digit can represent a number of ones or tens. The player decides which it will be based on the number they are counting on from and the goal of getting close to 300 at the end of their eight turns. Along the way, they practice counting on by tens or ones from a variety of numbers. A good way to introduce the game is to set a goal of 100. But such a low target limits counting by tens. You can adjust the game to any start number and any target, such as starting at 200 and playing *500 Is Perfect*, or starting at 400 and playing *3,000 Is Perfect* (in this version, the digit can represent tens or hundreds). Some variations are shown in the game board examples.

Materials: A *300 Is Perfect* game board (optional), a 10-sided die or digit cards

Directions:

1. Players take turns rolling a digit.
2. The player decides whether the digit will represent the number of ones or tens.
3. The player begins at 0 and counts on the amount that they determined.
4. On the next turn, the player rolls again and decides how to use the number.
5. They count on that amount from the new number.
6. The player must roll exact
7. The player closest to 300

The image on the left shows an example o you could modify it for a different starting by hundreds and tens with a goal of 3,000.

300 Is Perfect

Directions: Start at 0. Roll a number. Decide if that is the number of ones or tens to count on from 0. Roll again and count on from your new number. Get as close to 300 as possible without going over.

Tens	Ones	Total
6		60
	9	69
7		139
6		199
	4	203
	7	210
5		260
	8	268

online resources — This resource can be downloa

ACTIVITY 10.14

Name: Just One, Which One **Type:** Center

About the Center: This center is an independent practice to reinforce the notion of estimating a solution by adjusting one number in a problem (see Activity 10.4). It's a concept that students will need to practice, especially if their estimation experiences have been mostly procedural or focused on finding two benchmark numbers.

Materials: 10-sided dice, playing cards (queens = 0, aces = 1; remove tens, kings, and jacks), or digit cards; recording sheet (optional)

Directions:

1. Students generate a problem with dice or cards. Alternatively, you can provide problem cards written on index cards or printed on paper.
2. Students record the problem.
3. Students estimate the result by changing one number to a benchmark number and record the estimate.
4. Students estimate the result a second time by changing the second number.
5. Students identify which problem was easier for them to think about.

The recording sheet is shown. Students could fold a piece of paper in thirds, draw columns on a dry erase board, or write in their mathematics journals. Notice the extension added at the bottom of the recording sheet, asking them to explain whether it's easier to estimate by adjusting one or both numbers. You can swap that out with other prompts such as, "Tell me which problems were easier to estimate and why" or "Were there any problems that were easy for you to estimate if you changed either number? Which problems were they? What made them easier?"

Just One, Which One

Directions: Generate a problem and record it in the first column. Estimate the result by changing one number. Then estimate again by changing the other number. Star the one that's easier to think about.

Problem	Estimate #1	Estimate #2
77 + 57	80 + 57 = 137 ★	77 + 60 = 137

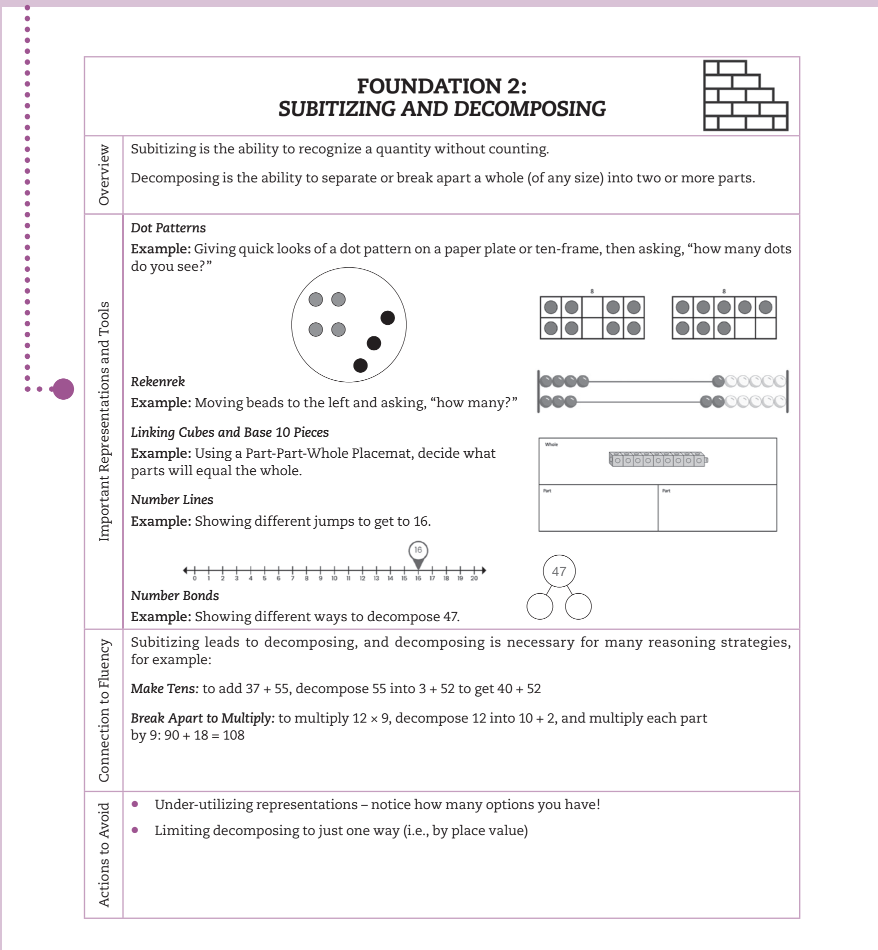
What are you noticing about which number to change to estimate?

online resources — This resource can be downloaded at **https://qrs.ly/psf6a5o**

Part 2 • Module 10 • Computational Estimation 219

Download the resources you need for each activity at this book's companion website.

Downloadable Foundation Briefs are quick-shot reminders that can be referenced while planning, teaching, and in discussion with colleagues and families.



FOUNDATION 2:
SUBITIZING AND DECOMPOSING

Overview

Subitizing is the ability to recognize a quantity without counting.

Decomposing is the ability to separate or break apart a whole (of any size) into two or more parts.

Important Representations and Tools

Dot Patterns
Example: Giving quick looks of a dot pattern on a paper plate or ten-frame, then asking, "how many dots do you see?"

Rekenrek
Example: Moving beads to the left and asking, "how many?"

Linking Cubes and Base 10 Pieces
Example: Using a Part-Part-Whole Placemat, decide what parts will equal the whole.

Number Lines
Example: Showing different jumps to get to 16.

Number Bonds
Example: Showing different ways to decompose 47.

Connection to Fluency

Subitizing leads to decomposing, and decomposing is necessary for many reasoning strategies, for example:

Make Tens: to add 37 + 55, decompose 55 into 3 + 52 to get 40 + 52

Break Apart to Multiply: to multiply 12 × 9, decompose 12 into 10 + 2, and multiply each part by 9: 90 + 18 = 108

Actions to Avoid

- Under-utilizing representations – notice how many options you have!
- Limiting decomposing to just one way (i.e., by place value)

FIGURING OUT Fluency

TEN FOUNDATIONS for Reasoning Strategies With Whole Numbers

A Classroom Companion

FIGURING OUT Fluency

TEN FOUNDATIONS for Reasoning Strategies With Whole Numbers

A Classroom Companion

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Contents

online resources

Visit the companion website at **https://qrs.ly/psf6a5o** for downloadable resources.

Preface

All students can develop procedural fluency, including fluency with basic facts, whole number operations, rational number procedures and operations, proportions, and solving algebraic equations.

A COMMITMENT TO FLUENCY

To ensure every student develops fluency, we must first

- understand what procedural fluency is (and what it isn't),
- respect fluency, and
- plan to explicitly teach and assess foundations for computational fluency.

If you have read our anchor book, *Figuring Out Fluency in Mathematics Teaching and Learning*—which we recommend in order to get the most out of this classroom companion—you'll remember an in-depth discussion of these topics. In fact, Chapter 3 of that book was titled "Good (and Necessary) Beginnings for Fluency." This book, therefore, is a more in-depth exploration of that topic—delving into those foundational skills that underlie all fluency work, whether it is for first instruction, practice, or intervention. Since publishing *Figuring Out Fluency in Mathematics Teaching and Learning* in 2021, we have realized that this topic needed and deserved a fuller discussion and more tools for you as a teacher than could fit in a single chapter. We came to understand that all the good reasoning strategies in the world don't really help students if there are not some key ideas and skills solidly in place for them. Read on and we'll explain what we mean.

WHAT PROCEDURAL FLUENCY IS AND IS NOT

Before we get into the heart of this book, it's important to briefly discuss what fluency *is* and what it *is not*. Similar to fluency with language, wherein you decide how you want to communicate an idea, fluency in mathematics is a decision-making process: As you look at the numbers in the problem, certain strategies make the most sense. Thus, fluency is about having flexibility and efficiency. If a student can "fluently add two-digit numbers," they will likely add 49 + 48 and 51 + 14 differently; for example, they might add 49 + 48 by thinking they are both close to 50, so they would mentally round up by adding 1 to 49, adding 2 to 48 (so 3 total), then add 50 + 50 and subtract out 3. It equals 97. The other problem might be solved by adding tens and ones or using a Jump-Up strategy. Each of these options are efficient and are a good fit for the problem. In neither case is a standard algorithm a good option. It is slower and involves more steps. Thus, being fluent is not equivalent to being adept at using an algorithm. That is called, well, being skillful with an algorithm. Fluency includes being adept at algorithms, but that is only one of a repertoire of reasoning strategies one can use, depending on the situation. Having this fluency requires strong foundations in number relations, decomposing, and the properties. Fluency is not about speed, though it is about efficiency. Take for example, the problem 403 – 299. A fluent student will take the time to first analyze the problem, think of a reasonable method, notice that both numbers are close to hundreds, and use that information to solve it efficiently (for example, a Count Up strategy or a Compensation strategy). A non-fluent student with skill using the standard algorithm will stack, regroup, and solve, moving quickly through the steps. This student is fast but not efficient, as the algorithm will take longer and involves more steps. The National Council of Teachers of Mathematics (NCTM) Procedural Fluency Position Statement (2023) is an excellent description not only of what procedural fluency is but also what is necessary to ensure all students develop procedural fluency.

RESPECT FLUENCY

We are strong advocates for conceptual understanding. We all must be. But there is not a choice here. Procedural fluency relies on conceptual understanding, but conceptual understanding alone cannot help students fluently navigate computational situations. They go together and must be connected. The foundations in this book connect conceptual understanding to the essential skills needed for employing reasoning strategies with the operations and beyond.

EXPLICITLY TEACH AND ASSESS FOUNDATIONS

We cannot overstate the importance of readiness in developing fluency. Readiness for procedural fluency means developing the concepts and skills needed in reasoning. In numerous research reports, using learning progressions or trajectories are found to have a positive impact on student learning. As an example, consider the readiness to apply the Making 10 reasoning strategy. Students are ready to learn this strategy when they understand these foundations:

- Commutative Property

  For 5 + 8, the thinking begins with an add on to the larger number (8 + 5).

- Distance to 10/Combinations of 10

  8 is 2 away from 10.

- Decomposing

  Decompose 5 so that it is 2 and some more (2 + 3).

- Associative Properties

  Reassociate the 2 from the 5 with the 8 [8 + (2 + 3) = (8 + 2) + 3].

- Adding 10 and Some More

  Add 10 + 3.

This may look like a long list, but these are number relations and concepts that become automatic with adequate time and experiences. It is simply broken out here to illuminate how important foundations are to being able to apply reasoning strategies.

So, what are the significant reasoning strategies for which we need these foundations? In *Figuring Out Fluency in Mathematics,* we introduced seven such strategies for computational fluency:

1. Count On/Count Back (Addition and Subtraction)
2. Make Tens (Addition)
3. Use Partials (Addition, Subtraction, Multiplication, and Division)
4. Break Apart to Multiply (Multiplication)
5. Halve and Double (Multiplication)
6. Compensation (Addition, Subtraction, and Multiplication)
7. Use an Inverse Relationship (Subtraction and Division)

Notice that while there are seven strategies in the complete list, there are no more than four for any particular operation. So, for a child to be fluent in subtraction, for example, they would understand, be able to use, and know when to choose each of the strategies that are useful for subtraction: Count Back, Use Partials, Compensation, and Use an Inverse Relationship (i.e., Think Addition/Count Up).

To be ready to learn these different reasoning strategies, though, we must teach and assess certain critical foundations. The alternative (teaching Make Tens when students don't have these necessary foundations) results in students not being able to enact the strategy, and thus they are stuck using a counting method or memorizing their facts. They haven't learned a significant strategy that is incredibly useful for computation with whole numbers and rational numbers. In other words, we have not provided students the necessary opportunities and experiences to develop fluency. In this book, then, we focus on those ten foundations that are necessary for developing fluency. We have created a module for each one so that teachers have a plethora of teaching, practicing, and assessing ideas to ensure students understand and are adept at using each foundation. They include the following:

- Number Relationships: Comparison and Estimation (Module 1)
- Subitizing and Decomposing (Module 2)
- Distance to 10, 100, and 1,000 (Module 3)
- Counting and Skip Counting (Module 4)
- Properties of Addition and the Inverse Relationship with Subtraction (Module 5)
- Properties of Multiplication and the Inverse Relationship with Division (Module 6)
- Multiplying by Tens and Hundreds (Module 7)
- Multiples and Factors (Module 8)
- Doubling and Halving (Module 9)
- Computational Estimation (Module 10)

USING THIS BOOK

This book can support your curriculum and other resources, adding to your collection of high quality, student-centered activities. Fluency foundations take time and repeated experiences to develop, so this book can be thought about as a pantry—open it up when you need an assessment prompt to gain insights into your students' thinking (assess); you are hoping for some ideas on stories and visuals that build the foundations (teach); or you need an engaging routine, game, or center to focus on a selected foundation (practice).

As mentioned, this book is a classroom companion book to *Figuring Out Fluency in Mathematics Teaching and Learning.* In that anchor book, we lay out what fluency is and barriers to a true focus on fluency, and we briefly discuss necessary foundations for fluency. We also propose the following:

- 12 "fluency fallacies" that clarify what fluency is and how to accomplish it
- 7 significant strategies across the operations (previously listed), all of which require these ten foundations
- 8 "automaticies" *beyond* automaticity with basic facts, several of which are addressed in this companion book (e.g., decomposing [breaking apart] numbers within ten, doubling, and halving)
- 5 ways to engage students in meaningful practice, including routines, games, and centers
- 4 assessment options that can replace (or at least complement) tests and that focus on real fluency
- 6+ ways to engage families in supporting their child's fluency

In Part 1 of this book, we highlight some of these big ideas to provide context for the fluency foundation modules. Part 1 is not a substitute for the anchor book but rather a brief revisiting of central ideas that serve as reminders of what was fully illustrated, explained, and justified in *Figuring Out Fluency in Mathematics Teaching and Learning*. Hopefully, you have had the chance to read and engage with that content with colleagues first, and then Part 1 will help you think about those ideas as they apply to the ten foundational ideas in this book.

Part 2 is focused on modules about teaching, practicing, and assessing each foundational idea. Each module includes the following:

- **Overview:** an overview for your reference and to share with students and colleagues
- **Assessment:** a list of prompts related to the foundation, along with a variety of quick checks to determine what a student knows related to the foundation
- **Explicit Instruction**: a series of teaching activities that incorporate manipulatives, representations, and student talk to help students make sense of the foundation
- **Quality Practice**: a series of practice activities, including routines, games, and center activities that engage students in meaningful and ongoing

practice to develop proficiency with the foundation while also serving as further opportunities to assess student learning

Note that most of the online resources include variations of the activities using numbers through the thousands place and decimals.

Pick and choose from Part 2! If these foundations are in your curriculum, then simply find the module that fits your needs and select the activities that best meet the needs of your students. Two of the needs you may identify are first instruction and intervention.

First Instruction: If your students are learning a fluency-related topic—for example, subtracting whole numbers or finding equivalent fractions—then ask yourself what foundations will be important to their success. Go to that module and select an assessment idea or practice activity to try out as a way to gain insights into students' readiness. You can immerse your students in a module, spending weeks exploring the foundation in depth, or you can identify a few activities to implement from time to time to ensure students maintain these necessary foundational skills. None of this has to happen all at once; activities can be woven into your instruction regularly over time.

Intervention: If you are providing intervention, then you are still picking and choosing content from across the modules. Looking through the book's table of contents, you might notice *decomposing* and wonder if the children you are working with are able to decompose numbers. Thus, you find an activity from this section and engage them in the task while you observe the extent to which they are able to decompose: Do they require manipulatives? Do they see all the ways or only some of the ways? Are they automatic? If the answer to the last question is "No," then continue to provide experiences for the child(ren), choosing more concrete activities if needed and moving into more abstract activities. If the answer to the automatic question is "Yes," then find another activity from another module and repeat.

Part 3 is about implementation! A series of FAQs are provided for implementation in various settings—the classroom, intervention setting, and at home. In addition, more ideas are provided for monitoring student success and ensuring that we continue to attend to students' emerging mathematics identities and agency as we engage in developing strong foundations.

WHO IS THIS BOOK FOR?

With over 120 instructional activities, 60 assessment prompts, and a companion website with resources ready to download, this book is designed to support many audiences, including classroom teachers, special education teachers, mathematics interventionists, tutors, and parents. The brief explanation of the foundation, suggestions for explicit strategy instruction, and range of options for practice make this resource useful for whole-class instruction, one-on-one instruction, and enjoyable mathematics at home. Additionally, those who lead teacher preparation programs can use this book to galvanize preservice teachers' understanding of the foundations necessary for fluency and provide these emerging teachers with a wealth of classroom-ready resources to use during internships and as they begin their career.

Figuring Out Fluency—Ten Foundations for Reasoning Strategies With Whole Numbers, A Classroom Companion is one of six companion books in the complete *Figuring Out Fluency* series. Each of these companions offers over 100 activities to support student reasoning related to different operations and types of numbers. We think of this book as sort of "Book 1.5." It is a useful in-between text between *Figuring Out Fluency in Mathematics Teaching and Learning* (the anchor book) and the other classroom companions:

Figuring Out Fluency—Addition and Subtraction With Whole Numbers

Figuring Out Fluency—Multiplication and Division With Whole Numbers

Figuring Out Fluency—Addition and Subtraction With Fractions and Decimals

Figuring Out Fluency—Multiplication and Division With Fractions and Decimals

Figuring Out Fluency—Operations With Rational Numbers and Algebraic Equations

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PART 1

FIGURING OUT FLUENCY FOUNDATIONS

Key Ideas

WHAT IS FLUENCY IN MATHEMATICS AND WHY IS IT IMPORTANT?

In construction, a foundation is the load-bearing part of a building. So it is with fluency—fluency is built upon foundational concepts and skills. Without such foundations, students are unable to build fluency with basic facts, whole numbers, and more. What makes a strong foundation is based on what one is trying to build. So, we start with the goal of procedural fluency. Try out these problems using any strategy that you like:

$$398 + 535$$

$$504 - 495$$

$$1{,}435 \div 7$$

How did you find the sum of the first example, the difference in the second, and the quotient in the third? Did you use strategies or algorithms? Did you start with one strategy and shift to another? Each of these problems can be solved efficiently using a strategy other than the standard algorithms. For example, in 1,435 ÷ 7, the dividend can be broken apart into 1,400 + 35, and each part can be divided by 7, resulting in 200 + 5. A person who demonstrates fluency with division notices the following:

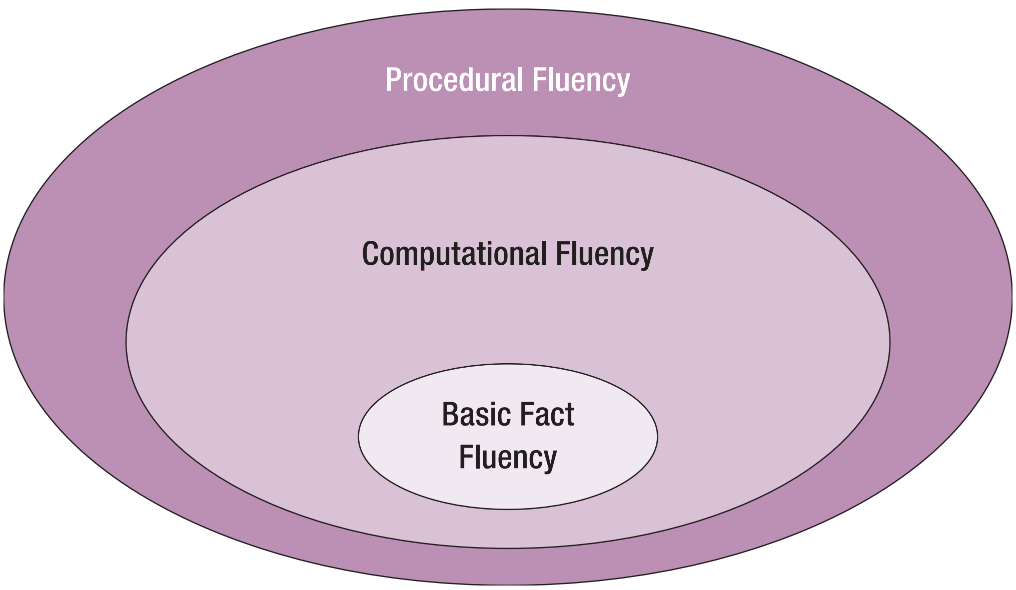
- The dividend (1,435) includes multiples of 7.

  Foundation: Knowing multiples (Module 8)
- The dividend can be decomposed into more noticeable multiples of 7 (1,400 + 35; not decomposed by place value).

  Foundation: Being able to flexibly decompose (Module 2)
- If 14 ÷ 7 = 2, then 1,400 ÷ 7 = 200.

  Foundation: Multiplying by tens and hundreds (Module 7)

FIGURE 1 ● The Relationship of Different Fluency Terms in Mathematics



Importantly, with these foundational concepts and skills in place, a person has access to a strategy that is more efficient and less error-prone than long division. Thus, these foundations are the necessary and good beginnings of fluency! Revisit the other problems posed above and ask yourself, "What foundational concepts and/or skills allow me to solve this problem more efficiently than using a standard algorithm?"

Procedural fluency is an umbrella term that includes basic fact fluency and computational fluency (see Figure 1).

Basic fact fluency attends to fluently adding, subtracting, multiplying, and dividing single-digit numbers (see Figure 2).

FIGURE 2 ● Basic Fact Strategies and Their Extensions

BASIC FACT STRATEGY	BASIC FACT (SINGLE DIGIT) EXAMPLE	EXTENSIONS TO OTHER TYPES OF NUMBERS
Making 10	7 + 9 = 6 + 10 = 16	97 + 35 = 100 + 32 3.9 + 1.4 = 4 + 1.3
Pretend-a-10 (Compensation)	9 + 6 → 10 + 6 → 16 16 - 1 = 15	3,499 + 5,148 → 3,500 + 5,148 – 1
ThinkAddition	11 - 7 → 7 + ? = 11	89 - 75 → 75 + ? = 89 $9\frac{1}{8} - 8\frac{1}{2} \rightarrow 8\frac{1}{2} + ? = 9\frac{1}{8}$
Doubling	4 × 7 = 2 × 7 × 2	$4 \times 2\frac{1}{2} = 2 \times 2\frac{1}{2} \times 2$ 5 × 28 = 5 × 2 + 14
Add-a-Group	6 × 7 = 5 × 7 + 7	26 × 4 = 25 × 4 × 4
Subtract-a-Group	9 × 8 = 10 × 8 - 8	99 × 8 = 100 × 8 - 8
Think Multiplication	45 ÷ 9 → 9 × ? = 45	14.35 ÷ 7 → 7 × ? = 14.35

Computational fluency refers to the fluency in four operations across number types (whole numbers, fractions, etc.), regardless of the magnitude of the number. Procedural fluency encompasses both basic fact fluency and computational fluency plus other procedures, such as finding equivalent fractions.

Procedural fluency is defined as solving procedures efficiently, flexibly, and accurately (National Council of Teachers of Mathematics [NCTM], 2014; National Research Council, 2001). The meaning of these three components are

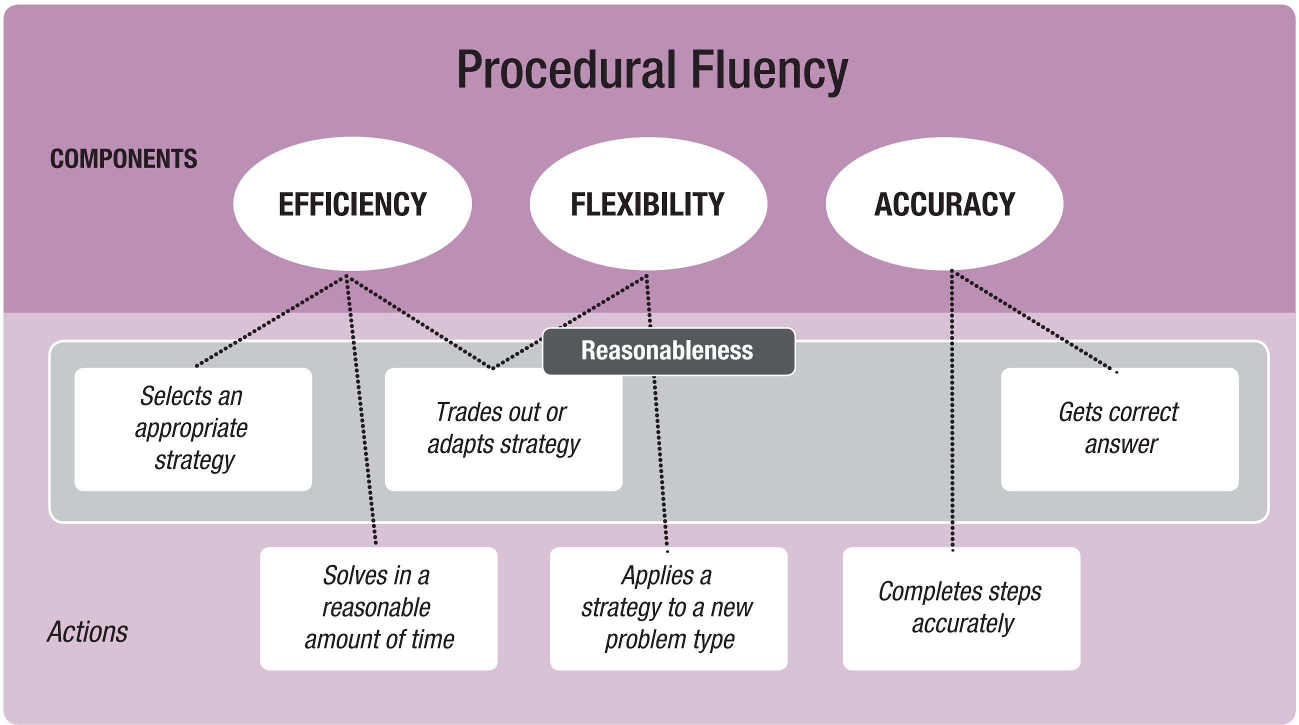
Efficiency: Solving a procedure in a reasonable amount of time by selecting an appropriate strategy and readily implementing that strategy.

Flexibility: Knowing multiple procedures and applying or adapting strategies to solve procedural problems (Baroody & Dowker, 2003; Star, 2005).

Accuracy: Correctly solving a procedure.

To focus on fluency, we need specific, observable actions that we can look for in order to assess what students are doing as they solve computational problems. We have identified six such actions. The three components and six fluency actions (and their relationships) are illustrated in Figure 3.

FIGURE 3 ● Procedural Fluency Components, Actions, and Checks for Reasonableness



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Three of the six fluency actions (should) attend to reasonableness. Fluency actions and reasonableness are described later in Part 1, but first, it is important to consider why this bigger (comprehensive) view of fluency matters. Real fluency is the ability to select efficient strategies; to adapt, modify, or change out strategies; and to find solutions with accuracy.

TEACHING TAKEAWAY

Real fluency is the ability to select efficient strategies; to adapt, modify, or change out strategies; and to find solutions with accuracy.

Real fluency is not the act of replicating someone else's steps or procedures for doing mathematics. It is the act of thinking, reasoning, and doing mathematics on one's own. The NCTM (2023) Procedural Fluency Position Statement describes what procedural fluency is and what is necessary to ensure all students develop procedural fluency, citing significant research along with instructional resources for classroom support.

WHAT DO FLUENCY ACTIONS LOOK LIKE FOR THE OPERATIONS?

The six fluency actions are observable and therefore provide insights into the foundational knowledge and skills students require. Each one is briefly described here, connected to the problems posed at the start of this section.

TEACHING TAKEAWAY

Selecting *an* appropriate strategy does not mean selecting *the* appropriate strategy. Many problems can be solved efficiently in more than one way.

FLUENCY ACTION 1: Select an Appropriate Strategy

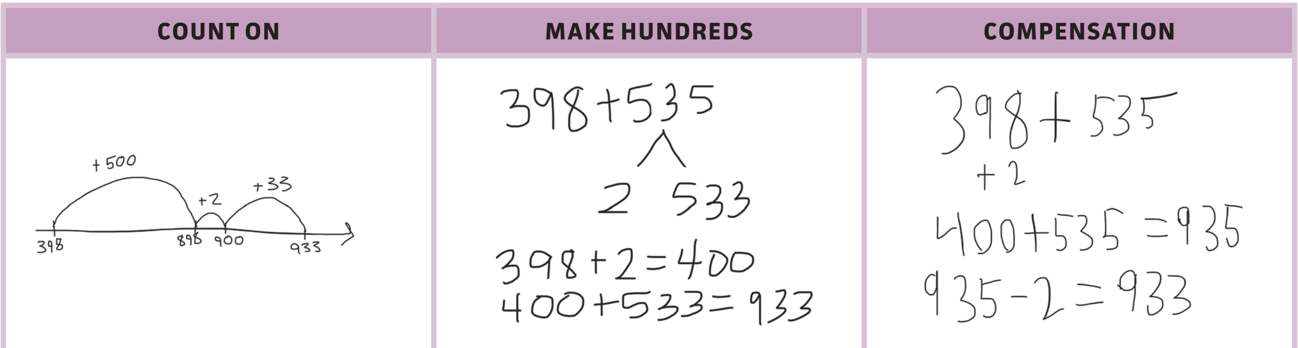
Selecting *an* appropriate strategy does not mean selecting *the* appropriate strategy. Many problems can be solved efficiently in more than one way.

Here is our operational definition:

> **Of the available strategies, the one the student opts to use gets to a solution in about as many steps and/or about as much time as other appropriate options.**

Consider 398 + 535. A student might start with 398, jumping up 500, then 2, and then 33 more (Count On strategy, see Figure 4). Another student might move 2 from 535 to 398 to make 400 and solve it (Make Hundreds strategy, see Figure 4). Or a student might leave 535 alone, add 400, and subtract 2 from their answer (Compensation strategy, see Figure 4). Each of these are appropriate for this problem because they each take about as many steps as the others. The standard algorithm, however, is not an appropriate choice, given the additional steps and time it would take to enact these addends.

FIGURE 4 ● Reasoning Strategies for Adding 398 + 535



Important points about these strategies include the following:

- They may be mental or written.
- They are flexible (there are other ways to use Count On, for example).
- The choice of a strategy requires fluency foundations—noticing that 398 is 2 away from 400, in this case (Make Hundreds strategy).
- The enactment of a strategy requires fluency foundations—decomposing and skip counting, for example, could be utilized for the Count On strategy.

FLUENCY ACTION 2: Solve in a Reasonable Amount of Time

The time it takes to solve a problem depends on the numbers in the problem and the mathematical maturity of the solver. A reasonable amount of time attends to two things: (1) the enactment of the selected strategy is efficient (e.g., Counting On in chunks rather than Counting On by ones) and (2) the solver works through their strategy without getting stuck or lost. For example, a student solving 504 – 495 may have noticed that they could use Think Addition (Count Up) and then drew a number line and counted by ones from 495 up to 504. This would be reasonable for a younger student learning subtraction as "find the difference," but with maturity, this strategy would be quicker, likely done mentally and by chunking the jumps (+5 to 500 and + 4 to 504).

FLUENCY ACTION 3: Trade Out or Adapt a Strategy

As strategies are better understood, students are able to adapt them or swap them out for another, more efficient strategy. For example, a student solving 1,435 ÷ 7 may first attempt to break apart the dividend by place value and get stuck because 1,000 is not a multiple of 7. Then, they decide to use Think Multiplication, reasoning that there are 100 sevens in 700—so 200 sevens in 1,400—and 5 sevens in 35, which add up to 205. When a strategy is not going well, a student goes back to other options, looking at the problem to see what might work. Similar to the original selection of a strategy, this is when a person relies on foundational understandings and skills to choose and enact a strategy.

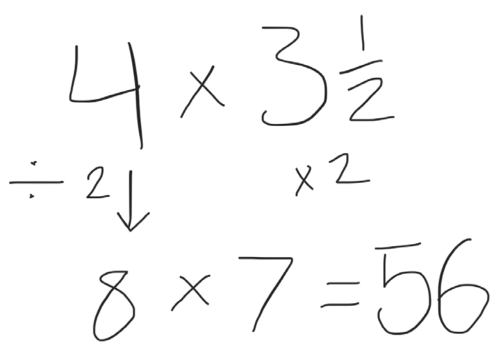
FLUENCY ACTION 4: Apply a Strategy to a New Problem Type

Take a strategy like compensation. It can be used with basic facts (e.g., thinking of 9 + 7 as 10 + 7 and take away 1), whole numbers (see Figure 4 above), and with fractions or decimals. Students generalize the idea that they can adjust a problem to make it easier to compute, and then they compensate to preserve equivalencies. Such generalizations are the properties in action!

FLUENCY ACTIONS 5 AND 6: Complete Steps Accurately and Get Correct Answers

An error at the end of a problem may be due to an error in how a strategy was enacted or due to an incidental error. For example, a student may think of the Halve and Double strategy accurately to solve $4 \times 3\frac{1}{2}$, as illustrated in Figure 5 by halving 4 and doubling $3\frac{1}{2}$, but make a computational error (doubling 4 instead of halving it).

FIGURE 5 ● Halve and Double Strategy Is Implemented Correctly, but a Computational Error Is Made



As these fluency actions indicate, true fluency requires decision-making, and those decisions require foundational understandings and essential skills. Procedural fluency is important for life and for higher-level mathematics. Most importantly, unrealized fluency creates significant barriers to students' productive and positive mathematics identity and agency. It all begins with ensuring students develop strong foundational understandings and skills!

REASONABLENESS

Reasonableness is more than checking your answer; it occurs in three of the six fluency actions as shown in Figure 3. Let's explore reasonableness for the problem 504 – 495. These two numbers are close together, thus Think Addition (Counting Up) is a reasonable strategy choice (Action 1). Carrying it out "reasonably" means to monitor if the selected strategy is going well (e.g., Count Up by Ones) and if not, to adapt it (e.g., count up by 5 to get to 500 and + 4 to get to 504) or trade out the strategy (Action 3). Finally, 9 is a reasonable answer because the numbers are close together (Action 6). At each of these phases, we see the role of foundations:

- To start, one must notice the relative size of the numbers.
- To chunk is to know the distance to 100.
- To know the answer is reasonable, one can use computational estimation (e.g., distance from 500).

Reasonableness is essential for fluency. The three Cs of Reasonableness (Choose, Change, Check) can provide strong support for students as they are thinking through a problem (see Figure 6). Importantly, a focus on reasonableness also supports the development of foundations and vice versa. For example, in asking, "Is this something I can do in my head?" a student will take time to look at the numbers in the problem and look for multiples or proximity to a benchmark, and the more they look for these relationships, the better they get at multiples or determining the distance to a benchmark.

FIGURE 6 ● Choose, Change, Check Reflection Card for Students

Checks for Reasonableness		
Choose	Change	Check
		
Is this something I can do in my head? What strategy makes sense for these numbers?	Is my strategy going well, or should I try a different approach? Does my answer so far seem reasonable?	Is my answer close to what I anticipated it might be? How might I check my answer?

Icon sources: Choose by iStock.com/Enis Aksoy; Change by iStock.com/Sigit Mulyo Utomo; Check by iStock.com/Indigo Diamond.

THE TEN FOUNDATIONS

Good and necessary beginnings—foundations—include the concepts and skills that are essential to reasoning. The example in the preface of 8 + 5 illustrates the key role of foundational understandings and skills to enact the Making 10 strategy. As another example, consider this multiplication problem: 49 × 15. Here is one way to solve this problem (which could be done mentally or in writing):

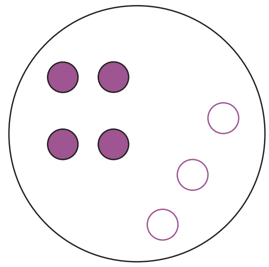
$$49 \times 10 = 490$$

$$49 \times 5 \text{ is half of } 490 \text{ (so } 245\text{)}$$

$$\begin{aligned} 49 \times 15 &= 490 + 245 \\ &= 500 + 235 \\ &= 735 \end{aligned}$$

Within this process, a student needs to understand the distributive property, be able to multiply by tens, find half of a number, and decompose to add. This one example clearly illustrates the critical need to ensure students have foundational knowledge and skills. So what are those good and necessary beginnings? Figure 7 provides an at-a-glance list of ten foundations that are necessary in giving students access to doing mathematics.

FIGURE 7 ● Foundations for Computational Fluency

FOUNDATIONS MODULES	WHAT IT IS (IN BRIEF)	EXAMPLE PROMPT
1. Number Relationships: Comparison and Estimation	Knowing approximately where a number is in relation to other numbers	Please place 28 on a number line. Use various endpoints [0, 50], [20, 30], or [0, 100].
2. Subitizing and Decomposing	Subitizing is visually seeing subsets of the whole to determine the whole; decomposing is starting with the whole and determining the parts	 How many dots? How do you see them? Decomposing: There are 10 turtles on two logs. How many might be on each log?
3. Distance to 10, 100, and 1,000	Seeing how far a number is from a benchmark (over or under)	How far is … • 8 from 10? • 37 from 40? • 107 from 100? • 288 from 300?
4. Counting and Skip Counting	Starting with a number and counting on or back by ones or other intervals (e.g., 20, 25)	Start at 48. Skip count by tens (or twenties). Start at 337 and count back 80.
5. Properties of Addition and Its Inverse Relationship With Subtraction	Using the commutative and associative properties of addition and using the relationship between addition and subtraction (i.e., if $a + b = c$, then $c - a = b$)	Give students an equation (for example, 25 + 15 = 40) and ask, "What other equations are true, using these same numbers?"
6. Properties of Multiplication and Its Inverse Relationship With Division	Using the commutative and associative properties of multiplication, the distributive property of addition over multiplication, and the relationship between multiplication and division (i.e., if $a \times b = c$, then $c \div a = b$)	Ask students, "Does 6 × 4 have the same answer as 4 × 6?" Then ask students to show/explain how they know it is true.
7. Multiplying by Tens and Hundreds	Being able to multiply any number (e.g., 16, 135, 5.2) by a multiple of 10 and know why it works Similarly understanding that 60 × 9 is the same as 6 × 9 × 10	Ask students to multiply 60 × 9. Ask, "How did you think about that?" Listen for answers that indicate understanding, not rule-based, incorrect explanations such as "I added a 0 on 54." Ask students to show 2,400 ÷ 6.

FOUNDATIONS MODULES	WHAT IT IS (IN BRIEF)	EXAMPLE PROMPT
8. Multiples and Factors	Recognizing when a basic fact is present (such as in 2,400 ÷ 6) and using that relationship to solve the problem	Give students two numbers (for example, 12 and 18) and ask what is alike and different about these numbers. If they don't focus on multiples and factors, then prompt for such responses.
9. Doubling and Halving	Being able to readily double or halve numbers (e.g., 48 or 250)	Ask students to double 36. If they are stuck, ask to double 30, then 6, and return to 36. Ask students to halve numbers with even digits (e.g., 264) and odd digits (e.g., 634).
10. Computational Estimation	Being able to quickly determine an answer close to the actual answer by using an estimation strategy (which does not include finding the exact answer and rounding it!)	Ask, "About how much is the answer?": 57 + 68 402 – 189 19 × 9 253 ÷ 6 Ask how they thought about it.

Each of these foundations are a full module in this book, which can be taught as a unit (replacing what might be in place for that foundation), used as a supplement (textbook coverage is often not sufficient to develop deep understanding and automaticity with these foundations), or used for interventions (because students who struggle with computation are often in need of more support with a foundational concept or skill).

For intervention, the use of these modules begins with figuring out which foundations are priority for the student. You have several options (using the table above) to decide where a student's strengths and needs are.

1. You can use the questions in Figure 7 to get a feel for what the student can do. Once you notice an area of need, stop there or move to a question they are likely to know well. It is counterproductive to go through a series of prompts wherein the student is struggling and experiencing stress or anxiety.
2. Go to the modules and read the assessment section at the beginning. There, you will find what you really need to be looking and listening for, along with six quick assessments (prompts) that lend to gaining insights into the students' understanding and skill.
3. Determine if the student needs to better understand the foundation conceptually. The first five activities in each module lean toward instruction for understanding the foundation. The remaining activities in a module provide opportunities for repetition. These are designed for students who show understanding but need more opportunities to work with the skill so that it becomes automatic and usable.

Part 3 helps you think about implementation. A series of frequently asked questions (FAQs) are provided for implementation in various settings—the classroom, the intervention setting, and at home. In addition, more ideas are provided for monitoring student success and ensuring that we continue to attend to students' emerging mathematics identities and agency as we engage in developing strong foundations within initial classroom instruction or intervention.

PRODUCTIVE BELIEFS ABOUT FLUENCY AND ITS FOUNDATIONS

With fluency defined, it is important to state that every student can develop procedural fluency. Attaining fluency for every student requires productive beliefs about fluency, described in Figure 8, which is also in our *Figuring Out Fluency* anchor book.

FIGURE 8 ● Productive Beliefs About Procedural Fluency

1. Procedural fluency is an attainable goal for each and every student. Each student is capable of developing a repertoire of strategies and learning skills at applying those strategies flexibly, efficiently, and accurately.
2. Procedural fluency is a function of opportunity, experience, and effort. Differentiated supports enable each and every student to understand and use a range of strategies.
3. Procedural fluency instruction is higher-order thinking, as students create strategies, generalize when to use a strategy, and explain why a strategy works. This increased level of thinking leads to greater understanding and performance for every student.
4. Every student must have access to instruction and resources that attend to all procedural fluency components and actions.
5. Having a range of ideas and strategies for solving procedures enriches everyone's learning. Therefore, every student benefits from heterogenous grouping; conversely, homogeneous grouping (ability grouping) is detrimental to developing procedural fluency.

Productive Belief #1 is easy to agree with but not easy to enact. What does it look like to engage students in ways that say to them, "You are capable. You can figure this out"? A start is to shift *away from* showing students how to do something and move *toward* students showing us how they did what they did. This shift requires more of a teacher, not less. This leads into Productive Belief #2: providing opportunities and experiences to support students' development of procedural fluency. Students need quality and substantial foundations in order to eventually develop fluency with an operation or procedure. In our *Figuring Out Fluency* anchor book, we describe these as Good (and Necessary) Beginnings for Fluency (Chapter 3). In that chapter, we describe conceptual understandings, properties, utilities, and skills that enable students to reason with basic facts and beyond. The *Figuring Out Fluency* companion books (see list in the preface) focus on developing fluency (e.g., with whole-number addition and subtraction), yet these books could not fully take on the readiness skills to set students up for success—that is the purpose of this book! *Figuring Out Fluency—Ten Foundations for Reasoning Strategies With Whole Numbers* lands squarely on the foundations students need in order to have success with the strategies developed in the other books.

HOW DOES CONCEPTUAL UNDERSTANDING DEVELOP STRONG FOUNDATIONS (AND FLUENCY)?

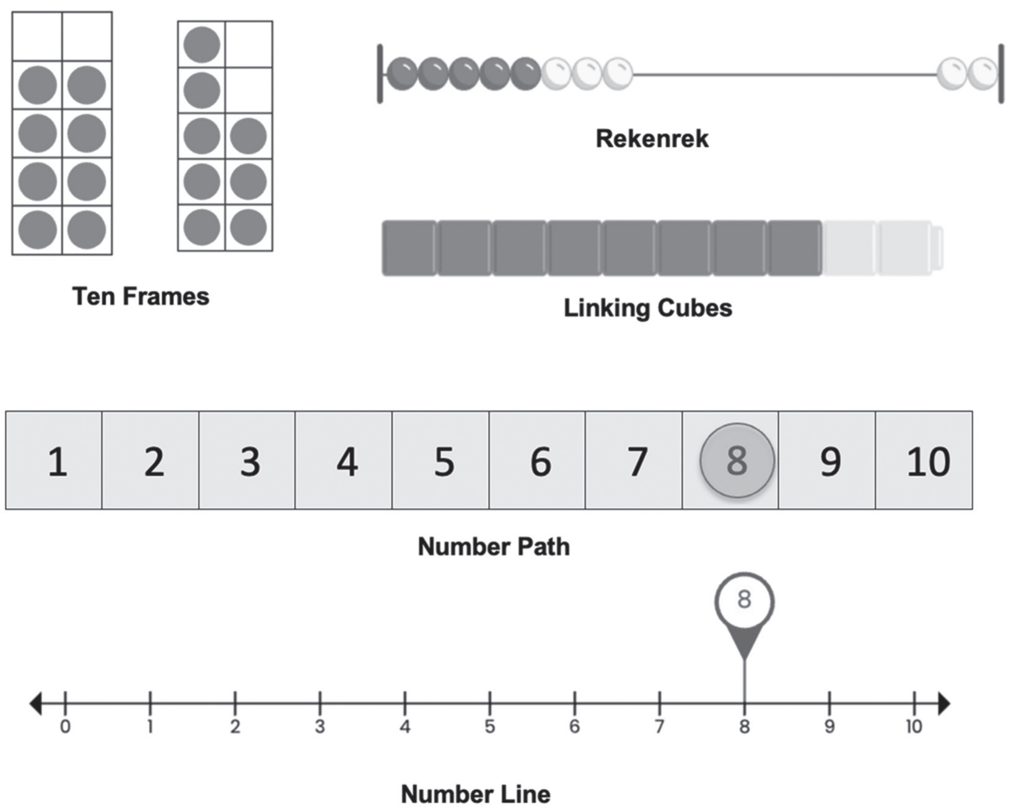
Conceptual understanding is connected knowledge: "mental connections among mathematical facts, procedures, and ideas" (Hiebert & Grouws, 2007, p. 380). For a subtraction problem, for example, conceptual understanding includes knowing the relative size of the numbers and understanding that subtraction can be interpreted as "find the difference" or "take away," that there are various strategies for finding answers to subtraction problems, and the various ways to represent those problems with manipulatives or drawings.

The NCTM offers this declaration related to the relationship between concepts and procedures: "Conceptual understanding must precede and coincide with instruction on procedures" (2023, p. 2). The elaboration explains that learning is supported when instruction on procedures and concepts is explicitly connected and iterative. Conceptual foundations lead to opportunities to develop reasoning strategies, which in turn deepens conceptual understanding. This is consistent with the classic concrete—semi-concrete—abstract (CSA) sequence (Bruner & Kennedy, 1965; Flores et al., 2018; Griffin et al., 2014). The CSA model is not a linear progression either. By design, the intent is to loop back to the C and the S to make sense of the A. Here, we highlight important ways to help students make connections and thereby develop strong foundations for fluency.

TOOLS AND REPRESENTATIONS

Manipulatives and visuals make mathematical relationships visible so that students can internalize abstract concepts. For example, the representations in Figure 9 help students see the relationship among numbers as well as the relative size of a number.

FIGURE 9 ● Representations to Show the Relative Size of the Number 8

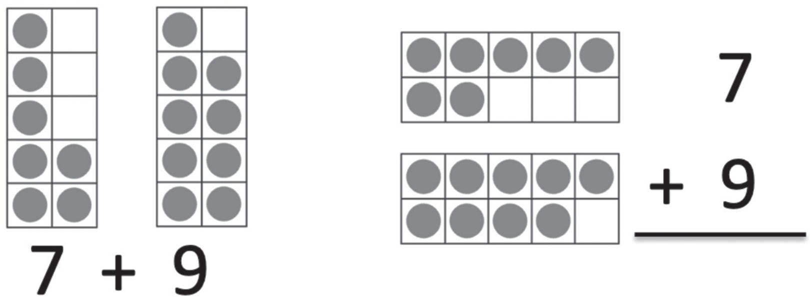


There is no shortage of objects that can be used to count. These objects can be used for subitizing, decomposing, exploring part–part–whole, and much more. Many tools, such as the ten frame, can initially be concrete (wherein students physically move counters on and off ten frames to illustrate numbers) then become visuals (similar to that pictured in Figure 10). And they may become mental images to support abstract reasoning. Here, we share four commonly used representations, offering some insights about the tool and how to use it.

TEN FRAMES

Ten frames can be used for subitizing, decomposing, number combinations, and implementing reasoning strategies such as Make Tens. A full row can be filled first to illustrate a number's relationship to 5, but it can be filled any way you choose to highlight a number relationship. Ten frames can be presented vertically or horizontally (as illustrated in Figure 10). A vertical orientation aligns with expressions written horizontally and, conversely, a horizontal orientation fits with a problem that is stacked.

FIGURE 10 ● Representing Addition on Ten Frames

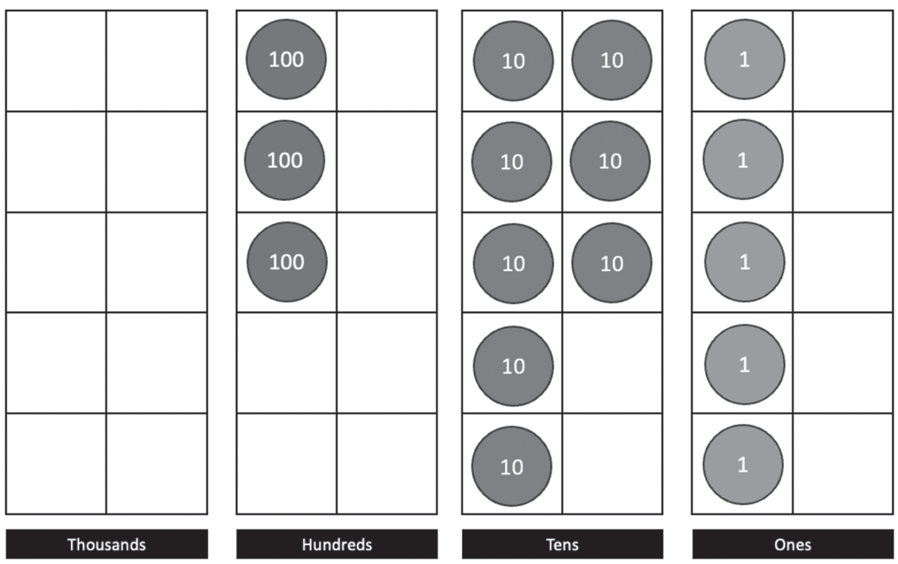


In both cases, we see how connections can be made among facts, ideas, and procedures.

PLACE-VALUE DISKS WITH TEN FRAMES

Place-value disks are nonproportional representations of numbers (see Figure 11). There are disks for ones, tens, hundreds, and so on. Pair them with ten frames for powerful representations of multi-digit numbers that can help students see how ones, tens, and hundreds can be regrouped.

FIGURE 11 ● Place-Value Disks and Ten Frames Showing 385



NUMBER PATHS AND NUMBER LINES

There is strong evidence that using a number line facilitates learning concepts and procedures with grade-level content and future learning (Fuchs et al., 2021). Being able to place numbers on a number line predicts student success years later (Geary, 2011). Number paths help students see quantity while also seeing how far numbers are from 0 or 10. They serve as an excellent tool for helping make sense of the abstract number line. Number paths and lines can also be positioned vertically and horizontally. In fact, if students are representing a situation that is vertical, then a vertical number line makes more sense. For example, if students are comparing heights of cubes or plants, then a vertical number line makes sense. Open number lines are particularly useful in reasoning because students do not need to get bogged down in the accuracy of unit lengths but they rather approximate the lengths to illustrate their thinking.

BOTTOM-UP HUNDRED CHART

The hundred chart is commonplace in Grade 1, 2, and 3 classrooms, yet its classic orientation is upside down (Bay-Williams & Fletcher, 2017). Figure 12 displays the Bottom-Up Hundred Chart. In this position, values go *up* on the chart as they *increase* quantitatively. This idea can and should be extended to decimal charts as well. Not only is this a stronger connection to the operations, but it is also more like the coordinate axis. A hundred chart helps students see place-value concepts and develop relational understanding (seeing that 48 is close to 50 and 10 away from 58). A hundred chart can also be cut in rows and taped together to form a 100 number path, which is an excellent bridge to working with number lines.

FIGURE 12 ● Bottom-Up Hundred Chart

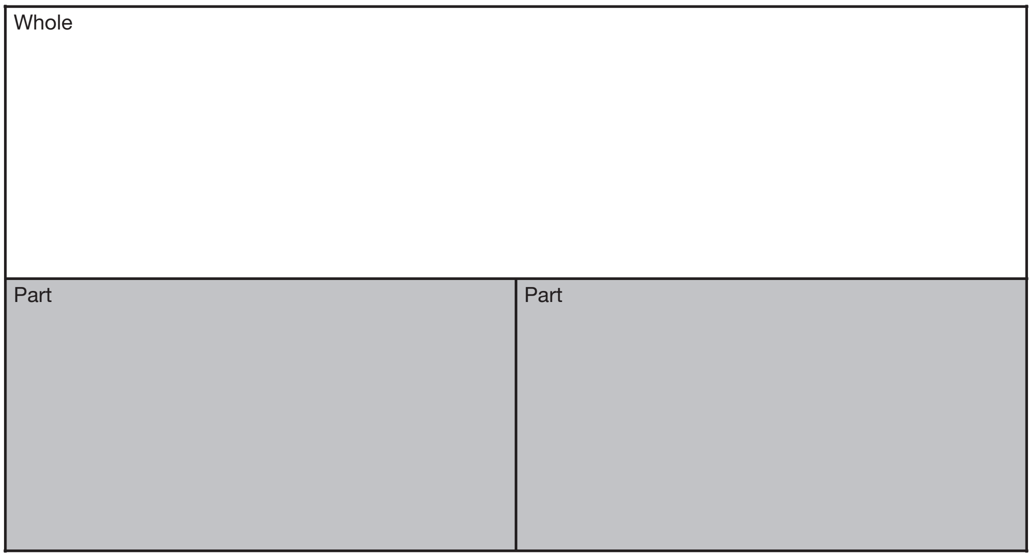
91	92	93	94	95	96	97	98	99	100
81	82	83	84	85	86	87	88	89	90
71	72	73	74	75	76	77	78	79	80
61	62	63	64	65	66	67	68	69	70
51	52	53	54	55	56	57	58	59	60
41	42	43	44	45	46	47	48	49	50
31	32	33	34	35	36	37	38	39	40
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

online resources This resource can be downloaded at **https://qrs.ly/psf6a5o**

PARTS AND WHOLE RECTANGLE

Addition is the joining of parts to make a whole, and subtraction is seeing the difference between a whole and a part. Thus, the part–part–whole graphic (Figure 13) provides a layout that illustrates this relationship, helping students see quantitative relationships, and can be used to place manipulatives such as linking cubes (see Activity 2.2), counters, or base ten pieces. Students can be given the whole and asked to find one or both parts or be given parts and asked to find the whole.

FIGURE 13 ● Part–Part–Whole Placemat



This resource can be downloaded at **https://qrs.ly/psf6a5o**

Eventually, students can replace objects with numbers: for example, placing numbers from a story problem on separate sticky notes and deciding where to place them to represent the story. This is true with whole numbers, fractions, or decimals. While many parts and whole visuals have two parts, they could have three or more parts, which simply involves adapting the part–part–whole placemat template (which is available in the downloadable slides). It can also be positioned vertically or horizontally.

MATHEMATICAL LANGUAGE

Similar to number lines, there is strong evidence that supporting students' use of mathematical language will support their learning of mathematics (Fuchs et al., 2021). Mathematics is based on very precise language, where meanings of words can change in different circumstances (e.g., *increased by* and *multiply by*) and where one word can make a difference in what operation is needed (e.g., *How many?* versus *How many more?* or *How many more?* versus *How many times more?*). As we listen to students describe their foundational concepts or skills, we need our own skills to help students develop this precision. Asking students to restate or rephrase can help teachers assess understanding while helping students learn the language of mathematics (Chapin et al., 2013). Example prompts include the following:

- "So, you said …" [revoice, inserting more precise language]
- "You used the hundred chart and counted on …?" [paraphrase using mathematical language]

- “Please repeat what you/someone just said.” [listen for precise language and understanding]
- “Explain ____ using the words _____ and ____ in your explanation.” (e.g., “How might you explain what you did with the linking cubes using the word *decompose*?”)

It’s also powerful to have students revoice their thinking while they practice and as they play games with partners, as it helps them

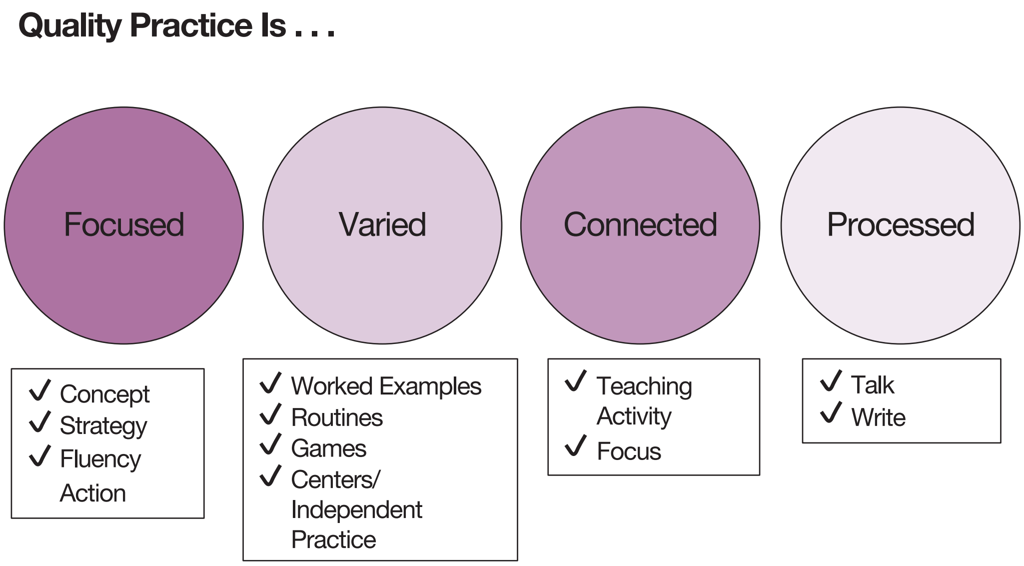
- clarify their process by hearing their thinking,
- strengthen metacognition and retention,
- provide another practice exposure to their partner, and
- gain insight into someone else’s strategy or process.

A second element to mathematical language is the broader goal of getting students to talk (classroom discourse). The moves described above support this goal as well. Think-alouds and peer tutoring are consistently found to positively impact student learning. And when students are talking about their thinking, teachers are getting a much stronger sense of what the student understands. Think—pair—share is an age-old yet underutilized classroom practice that ensures all students have processing time, are expected to articulate their thinking, and have the opportunity to learn from others.

WHAT DOES QUALITY PRACTICE LOOK LIKE FOR THE FOUNDATIONS?

Fluency practice is not a worksheet! This is the title of Chapter 6 in our anchor book, *Figuring Out Fluency in Mathematics Teaching and Learning*. Worksheets do not support good beginnings! Figure 14 provides a visual to capture the elements of quality practice, with example tasks within each.

FIGURE 14 ● Quality Practice Is Not a Worksheet!



This is not to say that practice requires a lot of time—quite the contrary. Short but ongoing, engaging practice is what is needed. If practice is too long, it can become unproductive. If it is too short, students don't get enough practice to solidify their understanding or develop the automaticity they need with the skills. The right amount of time gives enough exposure and keeps students engaged. There is no set number of minutes. It varies from practice activity to student to topic.

ROUTINES

A routine is a familiar, adaptable protocol for engaging students in learning through thinking and discussion. Many routines also use representations. Thus, routines help students build conceptual understanding and make connections to strengthen their emerging foundational concepts and skills. Routines can foster positive mathematics relationships within the classroom community (Berry, 2018). The exchange of ideas during a routine is essential for advancing student understanding and fluency. Discussion within routines reassures students that their emerging, possibly less common strategies are viable and used by others. Learners build confidence when they see that their strategy is reasonable and taken up by others.

Every module has several routines that focus on the selected foundation. But these routines are often adaptable to other foundations or to reasoning strategies. The keys to using routines effectively are to do the following:

1. Ensure all students understand the purpose of the routine and practice the steps of the routine.
2. Allow individual think time.
3. Make space for partner work.
4. Conduct full-group discussions of ideas.
5. Pose questions that focus on the key mathematical goal of the routine.
6. Encourage multiple representations and ideas.
7. Resist injecting your ideas or approaches too soon (or at all).
8. Keep routines short (5 to 10 minutes).

Try the routines but don't stop after one attempt. It is typically in the fourth or fifth round that students anticipate what is happening and the routine becomes more productive for everyone. Routines can be adapted—use fewer or more problems, change the steps, or trade out the representation.

GAMES

Games offer enjoyable practice. That joy is a benefit, not a rationale. The reason games are quality practice is because, similar to routines, they engage students in talking about their reasoning and listening to each other's strategies and provide an opportunity for teachers to listen and formatively assess. If you were to tally all the problems a student solves when playing a game, the list would certainly fill a worksheet. So, games provide opportunities for substantial and meaningful practice … at least, they *should*. Games that have a time component rob students of their processing time, add stress, and communicate that being good at mathematics means being fast. Games should not be timed nor should they pit students against each other in attempting to solve the same problem most quickly.

TEACHING TAKEAWAY

Games should not be timed and should not have students solving the same problem.

Because games are fun, students can forget they are practicing a mathematical skill. Tips to maximize learning when playing games include the following:

1. Tell students the purpose of the game and have them tell it back to you.
2. Give students sentence frames to help them articulate their thinking.
3. Provide recording sheets for students to record the problems they encountered.
4. Have students revoice their thinking as they complete a turn.
5. At the end of the game, ask students to reflect on their growth related to the skill they were practicing as well as any insights they gained about the mathematical ideas.

Games, similar to routines, can always be adapted. You can adapt games yourself or ask students how they would like to adapt the game. Some adaptations can simplify or increase the mathematical challenge; others can increase/decrease the complexity of the game itself (which will vary based on the age of students).

CENTERS

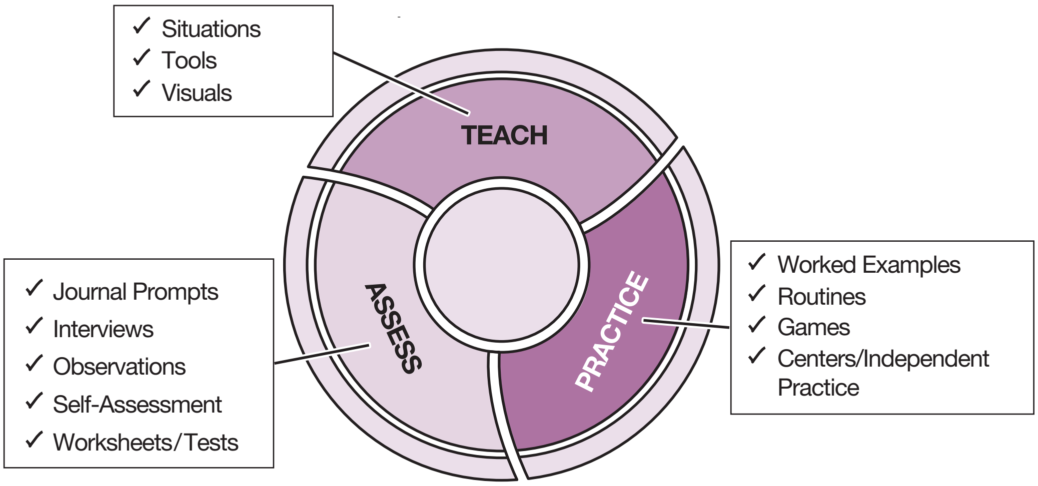
Centers are physical locations in the classroom set up with a mathematics activity that students can explore independently (alone or with a partner). Centers have traditionally been reserved for younger grades but are appropriate for all grades. Centers may have sorting tasks, choice problems, or games that can be played independently. Where routines and games provide opportunities for students to use language and learn with their peers, center activities provide extended time for individual engagement with a concept or skill. As students engage with the activity, they complete a recording page, providing themselves and you with a written record of their reasoning. Center activities can also be sent home or used as classroom activities. Students can work with partners or alone.

WHAT ARE THE RELATIONSHIPS AMONG TEACHING, PRACTICING, AND ASSESSING?

In Part 1, we have so far elaborated on what fluency means and thus what foundations become critical for students—in other words, the necessary and good beginnings. Throughout that discussion, representations and opportunities for students to use mathematical language took front and center stage. Where do representations and language use fit into the teaching, practicing, and assessing aspects of teaching? Everywhere! In fact, there is a lot of overlap and multi-purposing across teaching, practicing, *and* assessing activities. Students may be given a quick assessment for you to know where to focus instruction, but for them, it is also an opportunity to learn and to practice.

As you engage students in a game and require that they think aloud, you can assess their thinking as you manage the activity. The visual in Figure 15 illustrates that we use a variety of representations, visuals, and activities within each of these domains.

FIGURE 15 ● Ways to Teach, Practice, and Assess to Support Foundations and Fluency



This graphic can help you continue to vary the way in which students learn, practice, and are assessed. A list of all of the teaching and practice activities are provided in the Appendix. Keep your eye on the big ideas discussed in Part 1, and then flip to Part 3 for FAQs and ideas for implementing the modules in the classroom, during intervention, and in other settings.