What Your Colleagues Are Saying . . .

"I had an epiphany reading this book. I now really understand what fluency means when my students learn math. This book will help you teach strategies that will promote metacognition in your students. They will become confident and happy learners when dealing with math."

—Tamara Daugherty
Third-Grade Teacher
Orange County Public Schools, Orlando, FL

"You've heard the saying, 'You don't know what you don't know!' After reading *Figuring Out Fluency*, I found realization in this statement. After more than 30 years as a mathematics educator, I thought I knew everything there was to know about fluency. Wrong! This book is a must have for those who are novices and for those who want to know what they don't know about fluency."

—Thomasenia Lott Adams
Associate Dean for Research & Faculty Development,
University of Florida,
Gainesville, FL

"In my work as an elementary school teacher, math coach, and curriculum writer, fluency is always a hot topic of discussion in terms of how it develops and progresses across grade levels and grade spans. I truly appreciate the focus on conceptual understanding, reasonableness, and flexibility that is continually woven throughout every chapter of the book. These underpinnings alongside actionable ideas to use right away in classrooms make this book a valuable resource for any K–5 teacher or mathematics coach."

—Kristin Gray
Director K–5 Curriculum and Professional Learning,
Illustrative Mathematics

"Fluency is so much richer than facts and algorithms, and real fluency in mathematics includes reasoning and creativity. In *Figuring Out Fluency,* the authors take you on a journey of understanding, implementation, and reflection. They share relatable research, usable activities for the classroom and families, and most importantly the framework for an equitable action plan."

—Christine Percy
Florida Council of Teacher of Mathematics

"This book is an essential resource needed in every mathematics educators' hands! This is THE fluency playbook to ensure students engage in meaningful fluency learning."

—Crystal Lancour
Supervisor of Curriculum and Instruction,
Colonial School District,
Middletown, DE

"There is a piercing that readers will undoubtably feel at the many fallacies, unproductive beliefs, and inequities around fluency practiced in our classrooms today. *Figuring Out Fluency* is an incredible new book that offers mathematics educators everything they need to be equipped to create a coherent equitable approach to fluency. A MUST-READ!"

—Tara Fulton
District Mathematics Coordinator,
Crane School District,
Yuma, AZ

"I strongly appreciate that *Figuring Out Fluency* pushes us to think about fluency as providing learners opportunities to author their own ideas while developing flexibility in thinking and understanding facts, algorithms, and procedures. When learners have the authority to engage their own ideas, this positively impacts how they see themselves as doers of mathematics. *Figuring Out Fluency* challenges our conceptions of what it means to be fluent, and it unpacks ways for educators to support learners, families, and other educators to deepen their understanding. I particularly love the strategies provided in this book and the framing fluency."

—Robert Q. Berry, III
Samuel Braley Gray Professor of Mathematics Education,
University of Virginia,
Charlottesville, VA

"Are you ready to help your students connect their number talks and number routines to the real world? *Figuring Out Fluency* will give you the routines, games, protocols, and resources you need to help your students build their fluency in number sense (considering reasonableness, strategy selection, flexibility, and more). Our students deserve the opportunity to build a positive and confident math identity. We can help support them to build this identity by providing them with access to a variety of strategies and the confidence to know when to use them."

—Sarah Gat
Instructional Coach,
Upper Grand District School Board,
Guelph, Ontario, Canada

"In far too many settings and for far too many years, fluency has been considered as being adept at implementing computational algorithms. So, thank you Jennifer Bay-Williams and John SanGiovanni for this first deep analysis of the importance of fluency. Anchored by the components of fluency—efficiency, flexibility, and accuracy—this amazing resource, which is based on both research and classroom-validated instructional practice, fully addresses the absolute necessity of conceptual understanding of operations, the important role of properties, and student access to a repertoire of methods: Real fluency. This must-have resource will truly influence teaching and teacher education."

—Francis (Skip) Fennell
Professor of Education and Graduate and Professional Studies Emeritus,
McDaniel College,
Westminster, MD

"*Figuring Out Fluency* goes beyond other resources currently on the market. It not only provides a robust collection of strategies and routines for developing fluency but also pays critical attention to the ways teachers can empower each and every student as mathematical thinkers who can make strategic decisions about their computation approaches. If you are looking for instruction and assessment approaches for fluency that move beyond getting the right answer, this is the resource for you."

— Nicole Rigelman
Professor of Mathematics Education,
Portland State University,
Portland, OR

"Everything you need and want to know about fluency is clearly spelled out in Jennifer Bay-Williams and John SanGiovanni's masterful new book, *Figuring Out Fluency in Mathematics Teaching and Learning*! This incredibly amazing resource defines what fluency is along with specific actions teachers need to take to help students understand, choose, and use effective strategies. A must-have for all math coaches and every K–8 teacher, this book provides practical tools, great activities, and fun games! After reading this book, everyone will understand that mastery, fluency, and automaticity are just not the same thing!"

—Ruth Harbin Miles
University Adjunct,
Mathematics,
Mary Baldwin University,
Staunton, VA

"In this practical and comprehensive resource Jennifer Bay-Williams and John SanGiovanni take a deep dive into one of the often-misinterpreted components of rigorous mathematics instruction—procedural fluency. Along with thorough explanations, engaging mathematical routines, tasks, and games, the authors offer a 'just-right' amount of research to ground each of the claims to powerful instruction in mathematics. This is a timely good-for-all, necessary-for-some resource for teaching/learning in mathematics."

—Yana Ioffe
Former Elementary School Principal and
Preservice Faculty Advisor at Nipissing University,
North Bay, Ontario, Canada

"*Figuring Out Fluency in Mathematics Teaching and Learning* provides a masterful approach to unpacking the meaning of mathematical fluency while investigating widely held fallacies. The authors provide a plethora of high-cognitive demand activities that teach and develop fluency. These activities are sure to become my go-to resources for implicitly teaching fluency!"

—Melynee Naegele, President
Oklahoma Council of Teachers of Mathematics;
Moderator, #ElemMathChat; Educator Leadership Council,
EF+Math; Special Education Instructional Coach,
Claremore (OK) Public Schools

Figuring Out Fluency in Mathematics Teaching and Learning: The Book at a Glance

FIGURE 7.2 ● Reference Page of Reasoning Strategies and Automaticities

SEVEN SIGNIFICANT REASONING STRATEGIES	RELEVANT OPERATIONS
1. Count On/Count Back	Addition and subtraction
2. Make Tens	Addition
3. Use Partials	Addition, subtraction, multiplication, and division
4. Break Apart to Multiply	Multiplication
5. Halve and Double	Multiplication
6. Compensation	Addition, subtraction, and multiplication
7. Use an Inverse Relationship	Subtraction and division
AUTOMATICITIES	**RELEVANT OPERATIONS**
Basic facts	Addition, subtraction, multiplication, and division
Breaking apart all numbers through 10	Addition and subtraction
Base-10 combinations	Addition and subtraction
Using 25s	Multiplication and division
Using 15s and 30s	Multiplication and division
Doubling	Multiplication
Halving	Division
Fraction equivalents within fraction families	Addition, subtraction, multiplication, and division
Conversions between common decimals and fractions	Addition, subtraction, multiplication, and division

online resources This resource can be downloaded at **resources.corwin.com/figuringoutfluency.**

The book offers Seven Significant Strategies and other automaticities for building fluency in all number types from whole numbers to fractions, decimals, and integers.

Teaching Takeaways throughout the book help you recall important key ideas and highlight issues of access and equity.

TEACHING TAKEAWAY

Honoring strategies from other countries and cultures builds cultural relevance, strengthens the school–community partnership, and exposes students to more fluent thinking.

TEACHING TAKEAWAY

Students with disabilities benefit as much as other students from an instructional focus on fluency with efficiency, flexibility, and accuracy.

Stop and Reflect boxes throughout help you connect main ideas to your practice.

Stop & Reflect

The first five strategies involved ways to break numbers apart. Compensation instead involves imagining a simpler problem (and then adjusting it to preserve equivalence). How might you help students understand these two different ways of reasoning?

ACTIVITY 2.1
ROUTINE: "THAT ONE"

Materials: A short list of three or four expressions (see examples below)

Directions: Post the list of expressions you create. Have students identify which expression(s) would be solved most efficiently with a standard algorithm and which ones lend to a reasoning strategy. Have students explain their decisions.

GRADE 3 EXAMPLES	GRADE 4 EXAMPLES	GRADE 5 EXAMPLES	GRADE 6 EXAMPLES
• 99 + 14	• 302 – 199	• $5 \div \frac{1}{4}$	• 0.25 × 48
• 47 + 47	• 617 – 438	• $7 \div \frac{1}{3}$	• 9.89 × 12.3
• 23 + 67	• 933 – 750	• $3 \div \frac{1}{6}$	• 3.7 × 4.1

Thirty-six activities throughout the book take three flavors: routines, focus tasks, and games. Game boards and other student work mats are available for download at resources.corwin.com/figuringoutfluency.

ACTIVITY 4.1
FOCUS TASK: WHAT'S THE TEMPERATURE?

Materials: Visual of a thermometer (or a vertical number line), one per student or pair

online resources — Resources can be downloaded at **resources.corwin.com/figuringoutfluency**.

Directions: Explain to the students that you are going to give a clue, and they are going to tell you the temperature you are thinking of. Have students record the related equations. Examples include the following:

1. In the morning, it was 19 degrees, and then, it warmed up 20 degrees. What's the temperature? (19 + 20 = 39)
2. When you got to school, it was 67 degrees, but at recess time, it was 15 degrees cooler. What did the temperature drop to? (67 – 15 = 52)
3. It was 10 degrees when the sun set, and overnight the temperature dropped 18 degrees. What was the temperature in the morning? (10 – 18 = −8)

ACTIVITY 3.7
GAME: STAY OR GO

Materials: Bottom-Up Hundred Chart (see Figure 3.11), one per pair of students; deck of cards (remove all tens, jacks, and kings; queens = 0, aces = 1), one deck per pair; chip or marker for Hundred Chart

Directions: Place deck facedown between the partners. Both players take two cards and turn them over side by side to form two 2-digit numbers. The goal is for the partners to work together to estimate. Player 1 gives the front-end estimate, placing a marker on the appropriate place on the Hundred Chart. Player 2 looks at numbers in the ones place and says either "stay" or "go up one row" (moving chip, if needed). Students record their estimates on a recording sheet.

FIGURE 3.11 • Bottom-Up Hundred Chart

91	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**
81	82	83	84	85	86	87	88	89	**90**
71	72	73	74	75	76	77	78	79	**80**
61	62	63	64	65	66	67	68	69	**70**
51	52	53	54	55	56	57	58	59	**60**
41	42	43	44	45	46	47	48	49	**50**
31	32	33	34	35	36	37	38	39	**40**
21	22	23	24	25	26	27	28	29	**30**
11	12	13	14	15	16	17	18	19	**20**
1	2	3	4	5	6	7	8	9	**10**

Talk About It

1. How would you describe quality practice to a colleague?
2. What things might you look for in fluency practice?
3. What practice approaches should you keep doing? Which might you rethink?
4. How might you infuse the types of practice shared in this chapter (routines, worked examples, games, centers, and independent practice)?
5. How do you currently hold students accountable for practice? What new ideas might you also use?
6. How do you ensure that students reflect on what they learned through their practice?

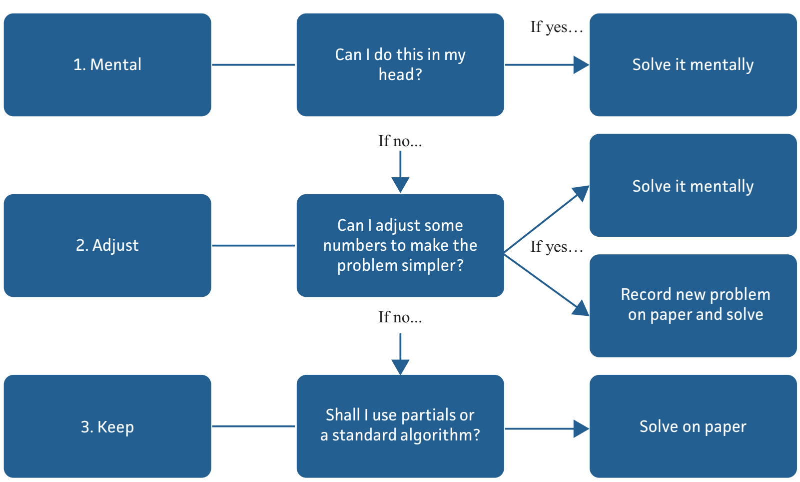
Act On It

1. **Review your practice resources**. Identify which of your practice resources meet the characteristics of high-quality practice. Identify which things you should consider modifying or purging. In other words, which aspects of fluency are practiced? How might you adapt or enhance practice in order to have a balanced approach across the components of fluency?
2. **Try an activity**. Identify one of the routines, games, or centers from this chapter (or any other chapter) and begin to work it into your mathematics practice regimen.
3. **Prepare worked examples**. For a topic that is important to your grade and/or is coming up soon, create a pair of worked examples for students to compare and discuss as part of or all of a lesson. Consider how you might use a worked example in a formative or summative assessment.

Talk About It and Act On It sections at the end of each chapter offer further discussion points and practical ideas for immediate implementation.

The book offers a handy metacognitive process chart to help students select a strategy for any given situation.

FIGURE 4.12 ● Metacognitive Process for Selecting a Strategy



FIGURING OUT Fluency IN MATHEMATICS

Teaching and Learning

Grades K–8

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Moving Beyond Basic
Facts and Memorization

Jennifer M. Bay-Williams
John J. SanGiovanni

FOREWORD BY Christina Tondevold

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Contents

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Visit the companion website at
resources.corwin.com/figuringoutfluency
for downloadable resources.

Foreword

When we talk about being "fluent in mathematics," what do we really mean? We know it's something we all want for our students, but do we know what it really looks like in practice? Does it look like a student being able to complete a set of flashcards in a certain amount of time? Is it quick and correct answers to basic addition or multiplication facts? Is it watching a child do mental mathematical gymnastics? In reality, real fluency is more complex, more nuanced, and actually more beautiful than that.

In this book, the authors describe their own journey toward understanding what fluency means and what it is—a journey that was neither straightforward nor direct. This resonated with me. My own trip to math fluency is similar to Jenny's and John's. My guess is that yours probably started off the same as well.

I grew up during a time of speed drills at the chalkboard, playing *Around-the-World* with flashcards, and staying in at recess if you didn't have your times tables down. I was good at memorizing facts, so I was good at all those drills. I also grew up at a time when teachers tended to stand in front of the board and show us how to solve problems using algorithms. I was good at mimicking, which helped me get the right answers on tests. Thanks to all of this, I thought I was good at math. I thought I was fluent.

When I started teaching, I continued the tradition of teaching-by-telling. I stood at the front of the classroom and made my students learn the procedures from the textbook. I gave timed tests. I used flashcards. I had my students pull out the numbers from a story problem and then use the key words I had asked them to memorize to figure out what operation to use. I made my students stay in at recess if they hadn't completed their "multiplication sundae." I thought I was helping them gain fluency.

The problem was that my definition of "fluent" was incorrect. By focusing on helping my students memorize and regurgitate the steps I had taught them, maybe I did help some kids get faster at getting answers. But what I also did was create the belief in their minds that math equals memorization; if you aren't good at memorizing, then you can't be good at math.

It wasn't until I was earning my master's degree that things changed for me. I was introduced to a "new way" to teach math. I put that in quotes because it was new to me, but it had been around for a long time. Even now, we hear people say "new math," but actually, people have been thinking about math in this "new" way for a really long time.

Once I was introduced to this new way, it changed everything for me. I started to see that I wasn't good at math—I was good at arithmetic. I was good at following directions that were laid out for me in the form of mathematical procedures, similar to how I can adeptly follow directions to build IKEA furniture. Being able to assemble a Billy Bookcase does not make me fluent at building furniture.

My own personal view of my ability to "do math" changed from that traditional view of math to one of being flexible with numbers, understanding why things worked, and being able to think through problems to get correct answers. This is what it means to be fluent in math.

Thinking back to our understanding about fluency—particularly, procedural fluency—what have we believed about fluency that's actually not true? What do students need to know to be considered procedurally fluent? How do we make practice fun and engaging? How do we assess fluency without using timed tests?

All those questions and more will be answered as you go through this book. These authors have spent their careers on a mission to help give consistent guidance on how to develop true fluency for students. My own change in how I teach math was greatly influenced by the work they have done (Jenny's work on the *Teaching Student-Centered Mathematics* books and John's work on the Howard County Math website, in particular). I'm excited that all their years of experience and research are coming together in this book to give practical advice that will get us all on the same path to building and sustaining fluency for our students.

So consider this book your travel guide for your trip to procedural fluency in math. Jenny and John lay it all out for us so that we know what this trip entails. It is a road map that can help us move beyond the path of doing worksheets and drills with only a focus on the answer. It gives us alternative pathways that focus on developing thinking and reasoning to get to that same destination.

This work can seem overwhelming, so Jenny and John make it manageable. They start at the beginning by showing some foundational understandings that students need and then share how those build into Seven Significant Strategies kids use when operating with numbers. These strategies go far beyond the basic facts we often associate with the word "fluency." Through the use of games, activities, and routines, the authors offer practical alternatives to worksheets and timed tests—actions that will truly help students build and assess their own fluency.

In this book, you will see how smooth and connected building fluency really is, from basic facts all the way through to fractions, decimals, and even algebra. You will find your head nodding in agreement. And as you put these ideas into practice, please recognize that it isn't easy, and it will take time. I've been trying to recover from my traditional ways of teaching math for 20 years now. It's hard to change something that is so deeply rooted, but it is so worth it.

It's time the narrative around fluency gets changed. You have the power to change it for yourself and for your students. Jenny and John guide you on the path to make that change through this book.

—Christina Tondevold

The Recovering Traditionalist

Preface

WHY THIS BOOK NOW?

We titled this book *Figuring Out Fluency* because, as a nation, an education system, and as educators ourselves, we have truly been trying for decades to figure out what mathematical fluency *really* means and ensure that this is the focus of instruction in our classrooms. Fluency has long been interpreted as adeptly implementing algorithms, yet real fluency is a creative process in which a person is able to choose a strategy that makes sense for the numbers at hand. *Real* fluency, therefore, requires conceptual understanding of the operations, understanding properties, and having a repertoire of methods. Compare these two metacognitive thought processes when encountering a computational problem like this: $\frac{3}{4} \times 24$

> *Student A*: What method am I supposed to use for these? Oh, it's multiplying fractions, so I put a 1 under the 24 and multiply the numerators and multiply the denominators.

> *Student B*: How can I find three-fourths of 24? Shall I do it mentally? Use a written method (which one)? Oh, hey, I can do this mentally—one-fourth of 24 is 6, so three-fourths is 18.

The procedure explained by Student A continues to be the most common way in which students (and adults) go about doing mathematics. Students who complete problems accurately are misdeemed "fluent." But *fluent* students would pursue the second line of thinking—noticing that they don't need the standard algorithm for this problem—and take the shortcut. Notice the lead-in lines here: *What method am I supposed to use* … versus *How can I find* … Teaching for fluency in mathematics (procedural fluency) focuses on the latter—helping students to become decision-makers, relying on their own thinking.

The field of mathematics education has come a long way in helping us accurately define procedural fluency and implement teaching that focuses on fluency. For example, *Adding It Up* (National Research Council, 2001) describes how dynamic procedural fluency really is:

> **Procedural fluency** refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently. (p. 121)

Adding It Up also speaks to the definition of procedural fluency: "Not all computational situations are alike. For example, applying a standard pencil-and-paper algorithm to find the result of every multiplication problem is neither necessary nor efficient" (2001, p. 122).

Additionally, The National Council of Teachers of Mathematics' (NCTM) *Principles to Actions* (2014) explains how effective mathematics teaching can support procedural fluency in one of its Effective Mathematics Teaching Practices:

> **Build fluency from conceptual understanding**: Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in *using procedures flexibly* as they solve contextual and mathematical problems. (p. 10; italics added)

So why do most students go about doing mathematics more like Student A rather than Student B? Here are a few possible reasons (the "we" refers to all of us in mathematics education, from textbook writers to teachers to policymakers):

- It is the way we learned and how we learned to teach.
- Procedural fluency and conceptual understanding are often seen as competing for attention, with "good" teaching attending to conceptual understanding.
- We misinterpret procedural fluency as a rigid mastery of algorithms (as the two metacognitive examples illustrate). Books and worksheets labeled as "fluency practice" are actually focused solely on mastery.
- We misinterpret phrases like "fluently multiply" to mean "use the standard algorithm."
- We want students to know at least one method, but often fail to go further, which actually denies access to fluency.
- We worry about using different methods for fear it will make students struggle and/or cause parent concerns.

We hope this list doesn't come across as deficit thinking, but we want to preface this book by clearly communicating that characterizing fluency only as mastering algorithms *is* a deficit view on fluency. In truth, fluency is so much richer than that. Fluency has been grossly oversimplified and therefore undertaught. *Real* fluency in mathematics involves reasoning and creativity. It varies by situation. Having fluency empowers students—shaping their positive mathematical identities and developing their sense of mathematical agency. Importantly, evaluating students based on an oversimplified (and inaccurate) perception of fluency—saying they are "good" at math because they are fast, or worse, saying they are "bad" at math because they are not—is a deficit view of *students*. Instead, fluency efforts must ensure that all students have access to a range of strategies and have regular opportunities to choose among those strategies.

One other possible reason that students are still functioning from a mastery perspective rather than a fluency perspective is there just is not enough support for teachers to shift toward a fluency approach. While many fantastic books provide guidance on developing conceptual foundations, far fewer attend to procedural fluency (even though, as noted earlier, this word is on many rote practice books). And that is why we wrote this book: to illuminate the meaning of real fluency, to help navigate the selection of strategies, and to provide a plethora of ideas to shift classroom practice toward a fluency approach.

WHO IS THIS BOOK FOR?

Our primary audience members are teachers and teacher leaders. These include novice and experienced K–8 teachers, mathematics coordinators, mathematics coaches, curriculum coordinators, mathematics teacher educators, professional development facilitators, and faculty in teacher preparation programs. Additionally, curriculum developers and policymakers who influence mathematics standards can benefit from the definitions created in this book and the many examples and activities.

WHAT WILL THIS BOOK DO FOR ME?

Our goal is that this book will give you the inspiration and the tools to think of fluency in a broader way and to value its importance alongside the conceptual understanding you have likely been working harder than ever to instill. In reading this book, you will

- Develop a deeper understanding of what procedural fluency is (and is not)
- Understand how to advantage students' understandings and skills to support their emerging fluency
- Learn which utilities, reasoning strategies, and automaticities to attend to in your teaching
- Have a robust collection of routines, games, and other activities that support a fluency agenda
- Develop techniques for assessing all components of fluency
- Be ready and excited to engage families in understanding and supporting fluency

ORGANIZATION

Figuring Out Fluency begins with figuring out what fluency means (Chapters 1 and 2). In the first chapter, we describe what fluency is (and why it is an equity issue), and in Chapter 2, we address the many fallacies that exist related to fluency. Many books attend to conceptual foundations for a particular type of number (e.g., fractions) or operation (addition and subtraction), with less attention to reaching procedural fluency. This book is the reverse. We condense the discussion of foundations to Chapter 3. Chapters 4 and 5 provide important lists for developing a fluency plan. Chapter 4 includes seven very useful strategies, which are useful across types of numbers—hence, we call them our Seven Significant Strategies. Chapter 5 identifies a list of procedures beyond the basic facts for which automaticity should be the goal. For example, being automatic with knowing combinations that equal 100 supports reasoning strategies. Ensuring every student develops these strategies and skills requires high-quality practice, the focus of Chapter 6. Chapter 7 provides a wide array of methods for assessing real fluency (attending to strategies, flexibility, and so on). With the confusion around what fluency really means, communicating with families is essential—and the focus of Chapter 8. We close with attention to planning in Chapter 9—putting all these pieces together so that you and your school/district will produce students who truly are fluent in mathematics.

Like developing fluency, teaching for fluency takes time and practice! There are many *teaching* strategies to consider—many issues to figure out. This book includes several features to support this "figuring out":

1. ***Activities.*** Thirty-six activities, including many routines and games, are integrated into the chapters. They focus on the often neglected components of fluency (strategy selection and flexibility), reasonableness, and connecting concepts to procedures.
2. ***Stop & Reflect prompts throughout the chapters.*** If you are reading alone, pause and think through these questions; if you are reading with others, stop and share at these points.
3. ***Teaching Takeaways*** are included in every chapter to highlight bold ideas to pay special attention to.
4. ***Talk About It*** questions are offered at the end of every chapter to revisit ideas proposed in the chapter. These can serve to guide book study discussions, as well as to help you process what you have read so that you can distill what you want to take away from the chapter.
5. ***Act On It*** suggestions follow the Talk About It section at the end of each chapter. Figuring out fluency requires taking action! Consider these a menu of ideas of how to get started. If you are able to use this book at the school level, these activities can be part of a faculty meeting or professional learning experience.
6. ***Resources*** are available online through the ***Figuring Out*** *Fluency* companion website to support your efforts. Each resource in the book that is also available for download is noted with the prompt: This resource can be downloaded at **resources.corwin.com/figuringoutfluency.**

And to help you continue to figure out fluency for specific content areas, this anchor book is complemented by **five *Classroom Companion* books**. These books elaborate on the pragmatic teaching ideas and activities in this anchor book and provide even more instructional and practice activities for use in the classroom. Working with a fantastic author team, Sherri Martinie, Rosalba McFadden, Jennifer Suh, and C. David Walters, the full set of *Figuring Out Fluency* will include these titles:

Figuring Out Fluency: Addition and Subtraction With Whole Numbers

Figuring Out Fluency: Multiplication and Division With Whole Numbers

Figuring Out Fluency: Addition and Subtraction With Fractions and Decimals

Figuring Out Fluency: Multiplication and Division With Fractions and Decimals

Figuring Out Fluency: Operations With Real Numbers and Algebraic Expressions

As we began this Preface, so we close. We have spent decades trying to figure out fluency. We are still on that journey. Our best thinking is shared in this book, and we welcome opportunities to continue the journey with you.

Acknowledgments

Just as there are many components to fluency, there are certainly many components to having a book like this come to fruition. The first component is the researchers and advocates who have defined procedural fluency and effective practices that support it. Research on student learning is hard work, as is defining effective teaching practices, and so we want to begin by acknowledging this work. We have learned from these scholars, and we ground our ideas in their findings. It is on their shoulders that we stand. Second are the teachers and their students who have taken up “real” fluency practices and shared their experiences with us. We would not have taken on this book had we not seen firsthand how a focus on procedural fluency in classrooms truly transforms students’ learning and shapes their mathematical identities. It is truly inspiring! Additionally, the testimonies from many teachers about their own learning experiences as students and as teachers helped crystalize for us the facts and fallacies in this book. A third component to bringing this book to fruition was the family support to allow us to actually do the work. We are both grateful to our family members—expressed in our personal statements that follow—who supported us 24/7 as we wrote during a pandemic.

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CHAPTER 1

What Does Fluency Really Mean, and Why Does It Matter?

The question posed by the title of this chapter is twofold. First, we have to understand all that is encompassed in mathematical fluency. In other words, we need to know what the *term* "fluency" means. Consider this list of terms and when and how they are used in the realm of mathematics education:

Fluency	Fluently	Procedural fluency	Computational fluency
Automaticity	Mastery	Know from memory	Memorize

Second, we have to consider how having or not having mathematical fluency impacts students and their futures. In other words, we need to understand what having fluency means to students *as human beings*. And it means a lot when we are talking about *real* fluency and not the memorization of facts and algorithms, which is often mistakenly referred to as fluency.

In this chapter, you will

- Define fluency through components and actions
- Establish fluency as an issue of access and equity
- Investigate games and routines that focus on real fluency in mathematics

WHAT IS FLUENCY IN MATHEMATICS?

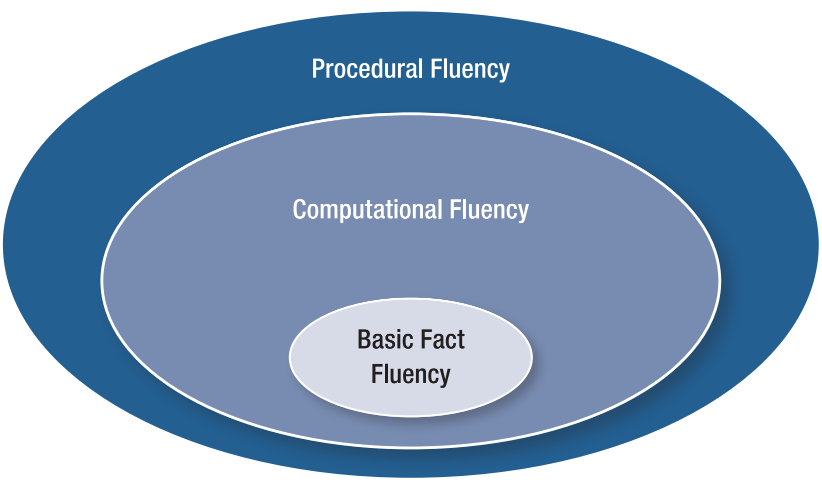
For a word used so often in mathematics standards and daily teacher conversation, it is surprising how often the word "fluency" is misused and misunderstood. In its simplest form, *Merriam-Webster* defines having fluency as being "capable of using a language easily and accurately."

If you have learned more than one language, you have experienced this meaning of fluency. When you are fluent in a language, you have a lot of options for how you might say something, like sharing what you did that day. When you are not fluent, you may have one way, and that way may be determined by the words you know.

What does it look like to use mathematics easily and accurately? Our world of math education has a much more complex relationship with the word and its definition. For example, compare the differences between the phrases “fluently use the standard algorithm” and “fluently add.” In the first case, “fluently” refers to being able to work through a process correctly and in a reasonable amount of time. But “fluently add” goes beyond implementing a procedure efficiently to knowing when that procedure is a good choice (and when it is not). Compare these two fluency phrases applied to the problem 99 + 45. In the first case, the expectation is that the student correctly employs the standard algorithm. In the second case, the expectation is that the student does not use the standard algorithm, instead noticing this is more efficiently solved using a strategy like Make Hundreds.

Consider two other terms on the chapter-opening list: procedural fluency and computational fluency. These two phrases differ only in their scope. Computational fluency refers to computation—the four operations. Procedural fluency includes more than the four operations, as there are many other procedures in mathematics—for example, comparing fractions, solving proportions, and simplifying expressions. And then there is basic fact fluency (fluency with operations involving single-digit numbers). Figure 1.1 illustrates the relationship among these phrases.

FIGURE 1.1 ● The Relationship of Different Fluency Terms in Mathematics



Procedural fluency is actually a term that has been well-defined for decades by such organizations as the National Research Council (NRC; Kilpatrick et al., 2001) and the National Council of Teachers of Mathematics (NCTM, 2014). Both organizations describe fluency as being able to apply procedures efficiently, flexibly, and accurately. To focus on real fluency, then, we need to understand what these constructs (adapted to nouns as efficiency, flexibility, and accuracy) *look like* as students are applying procedures. To begin, here is a brief explanation of what each is:

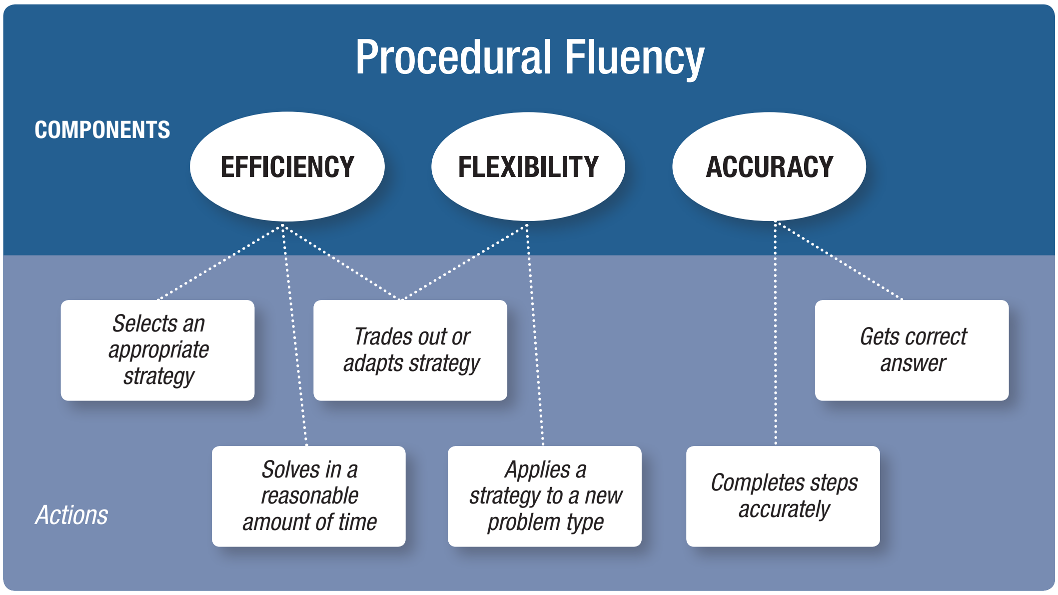
Efficiency: Solving a procedure in a reasonable amount of time by selecting an appropriate strategy and readily implementing that strategy.

Flexibility: Knowing multiple procedures and applying or adapting strategies to solve procedural problems (Baroody & Dowker, 2003; Star, 2005).

Accuracy: Correctly solving a procedure.

These components are also Big Ideas and include interrelated actions. More specific actions provide observable actions that can help to clarify what fluency really is. The diagram in Figure 1.2, expanded and adapted from Bay-Williams and Stokes-Levine (2017), provides an illustration of procedural fluency, connecting the three fluency components with six Fluency Actions.

FIGURE 1.2 ● Procedural Fluency Components and Related Fluency Actions



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FLUENCY ACTIONS

Take a look at the Fluency Actions represented in the lower section in Figure 1.2. Though some may seem straightforward, having a shared understanding of what fluency looks like in action is essential in working toward the goal of procedural fluency. Keep in mind that each of these actions is dependent on such things as grade level, experiences, and other contextual factors.

SELECTS AN APPROPRIATE STRATEGY

This phrase is commonly used, but not well defined. We use the following as a working definition of "appropriate strategy": *Of the available strategies, the one the student opts to use gets to a solution in about as many steps and/or about as much time as other appropriate options.*

For example, for the equation 412 – 297 = ?, one option is Think Addition (counting up from 297 to 300, to 400, then to 412) and another option is Compensation (subtracting 300, then adding 3 back on). Think Addition and Compensation can get to an answer in comparable time and steps, so both are appropriate strategies. *The standard algorithm* (regrouping) is not an appropriate strategy because it involves many more steps (and more time) than these other strategies.

This action develops through the understanding of different strategies, practice with each of those strategies, and practice selecting between strategies. We dig deeper into practice later in the book, but the game in Activity 1.1, *Just Right*, shows how students can practice selecting an appropriate strategy in a game context, matching a problem to an appropriate strategy.

ACTIVITY 1.1
GAME: *JUST RIGHT*

Materials: About 20 problems on cards; *Just Right* game board, one per pair

Directions: Players have a stack of about 20 cards that each have a problem on them. Players take turns flipping over a problem and deciding which strategy is most appropriate for solving it, putting their marker on that strategy of the *Just Right* game board (see Figure 1.3). For example, if an appropriate strategy to solve the problem is Compensation, the student describes how to solve the problem using compensation. Having correctly talked through the strategy, the player gets to place a marker on one of the Compensation spaces. The first player to place four markers in a row on the game board wins.

Note: In this example, the game focuses on both addition and subtraction. Problems to feature might include 302 – 297 = ___, 245 + 361 = ___, 500 + 97 = ___, and 499 + 237 = ___, among many others. This game board would be used after the strategies on the board have been taught, but the board can be modified to show only two or three strategies, focusing on addition only, for example. Students may be tempted to use inefficient strategies in order to get four in a row. To counter this, students can be required to record the equations and the strategy they used. *Just Right* can be played with decimals, fractions, and integers.

FIGURE 1.3 ● *Just Right* Game Board for Addition and Subtraction

JUST RIGHT

Directions: Flip over an expression. Decide which strategy is "just right" for the expression. Place a marker on the strategy. Be the first to get four markers in a row (horizontally, vertically, or diagonally).

Compensation	Count On/ Count Back	Make Tens (or Hundreds)	Partial Sums or Differences	Make Tens (or Hundreds)
Partial Sums or Differences	Think Addition	Compensation	Count On/ Count Back	Compensation
Count On/ Count Back	Standard Algorithm	Make Tens (or Hundreds)	Think Addition	Partial Sums or Differences
Standard Algorithm	Make Tens (or Hundreds)	Count On/ Count Back	Compensation	Think Addition
Compensation	Count On/ Count Back	Make Tens (or Hundreds)	Standard Algorithm	Partial Sums or Differences

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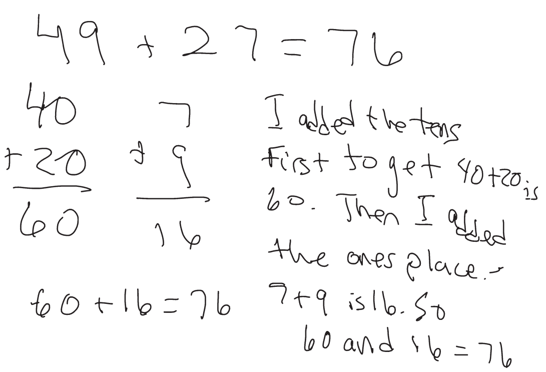
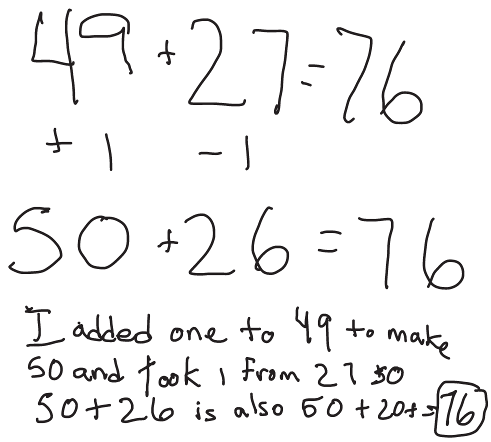
Selecting *an* appropriate strategy is not the same as selecting *the* appropriate strategy. Seldom is there only one appropriate strategy! For example, what are appropriate methods for 49 + 27? What are "inappropriate" or nonefficient strategies for 49 + 27? Here are some methods to consider (reflect on which ones *you think* are appropriate for a second grader and a sixth grader):

- Count On, skip counting by tens to 59 and 69, then counting by ones to 76
- Compensation, rounding up and subtracting the extra (50 + 30 – 4)
- Make Tens strategy, reimagining the expression as 50 + 26
- Partial Sums strategy, adding the tens and ones, then combining as 40 + 20 + 16
- Think of quarters (money) and reimagine the expression as 50 + 25 – 1 + 2
- Apply standard algorithm, adding the ones; regroup and add the tens

Take a look at how two second graders approached the same problem.





Both students use appropriate strategies (Partial Sums and Make Tens, respectively). Which appropriate strategy is used is an individual's preference. The efficiency of an appropriate strategy changes relative to the numbers and the individual. That is not to say that a preferred strategy—such as Counting On, Partial Sums, or Make Tens—is always appropriate. For example, Make Tens works well for a problem like 49 + 27 but loses efficiency (and therefore appropriateness) for problems like 336 + 237 or 1,378 + 756.

SOLVES IN A REASONABLE AMOUNT OF TIME

In general, a "reasonable amount of time" means moving through a process without getting stuck, lost, or bogged down. Time is not constant or definitive. For example, working through a fraction addition problem in three minutes might be appropriate in Grade 4, but not in Grade 7. And people naturally vary in the time they take to enact a strategy or algorithm.

TRADES OUT OR ADAPTS STRATEGY

Trading out and adapting are two separate actions, but we keep them together because they both refer to something fluent students do when their first choice of a strategy isn't working out for them. Trading out is going back to the start and making a new choice:

> Analea is solving 4 × 9 and selects doubling. She says, "Two times nine is 18," and then pauses. She doesn't know how to double 18. She starts over. "I know 10 fours is 40, so nine fours is 36."

Adapting a strategy is when a student adjusts the strategy while working, making it fit the numbers in the situation:

> Mantel is solving 549 ÷ 9 and decides to use a Break Apart strategy. He breaks apart into 500 + 40 + 9. Then, he realizes he lost the 54 he had spotted, so he rewrites the expression to be 540 + 9 and solves each partial quotient as 60 and 1.

APPLIES A STRATEGY TO A NEW PROBLEM TYPE

Let's say a student knows the Make Tens strategy with whole numbers. Applying that idea Make-a-Whole with decimals or fractions is using different types of problems. This action means the student has generalized a strategy. Sometimes, students overgeneralize—for example, applying an idea that works for addition to a subtraction problem. But this is part of the natural process of figuring out how and when various strategies work. Generalizations (and overgeneralizations) provide excellent opportunities for classroom discussions as students reason about when a strategy works, when it doesn't, and when it is useful.

Have you played *Scattergories*? This popular game has players give examples of a category (e.g., animal names) using a particular letter. In the fluency game *Strategories* (Activity 1.2), students actually generate examples of problems in which they would use a particular strategy from the game board.

TEACHING TAKEAWAY

Generalizations (and overgeneralizations) provide excellent opportunities for classroom discussions focused on when a strategy works, when it doesn't, and when it is useful.

ACTIVITY 1.2
GAME: *STRATEGORIES*

Materials: *Strategories* game card (see Figure 1.4), one per student

Directions: Pick an operation appropriate to your grade—for example, multiplication (the example shown here is for Grade 3 and up). Give each student a *Strategories* game card and provide think time for them to generate a problem for which they would use that strategy (alternatively, this can be a partner activity). For example, in this case encourage students to think of basic fact examples and then also bigger numbers. After each strategory has (at least) one example expression, pair students (or pair partners). Each person explains how to use the strategy for the problem they placed there. Collect class examples. Ask, "What do you notice about the problems in ____ strategory?" and "When is this strategy useful?" An alternate version poses an operation (e.g., addition) and a strategy (e.g., Make Tens), prompting students to generate different examples of it. Then, students might create examples like 29 + 9, 319 + 348, and 7.2 + 5.9 (depending on grade level and the type of numbers you have selected).

FIGURE 1.4 ● Sample *Strategories* Game Card

STRATEGORIES	
Adding a Group	
Subtracting a Group	
Near Squares	
Other Break Apart	

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Applying strategies to a new problem type is high-level thinking. The discussion following the collection of examples is critical to helping every student make connections between and across examples.

COMPLETES STEPS ACCURATELY

Accuracy is not just about the answer; it is about implementing an algorithm or strategy correctly. If students do not implement the process correctly, they will also get a wrong answer (usually). For a strategy such as Compensation, for example, a student knows that *for subtraction*, if they add 2 to the minuend, they do the same to the subtrahend (as compared to addition, wherein they would subtract 2 from the second addend if they added 2 to the first). For an algorithm, such as division, the student knows to find the greatest divisor for the front end of the dividend, and so on.

GETS CORRECT ANSWER

Students can complete steps correctly but make errors and therefore not get a correct answer. There are also times when there is more than one correct answer or the goal is an estimate or reasonable answer. Historically, this action has been one of two perceived indicators of fluency, with the other being "fast."

Collectively, these six Fluency Actions provide "observables" for the three components of fluency—efficiency, flexibility, and accuracy. Observables are and must be visible to students, families, and administrators. The more we can help every stakeholder see what fluency looks like in action, the better we can be at ensuring every child develops fluency. More on this in the assessment chapter.

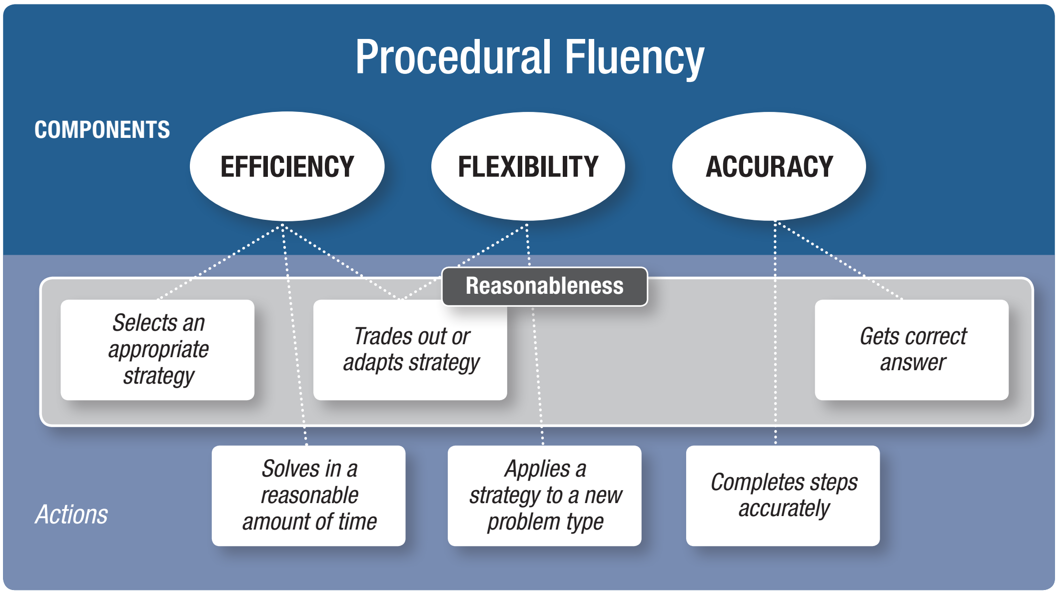
Stop & Reflect

Which Fluency Actions tend to be the focus of observations or assessments? Which Fluency Actions tend to be overlooked? How might the neglected actions become more visible to teachers and to students?

CHECKS FOR REASONABLENESS

Have you ever caught yourself trying to mentally solve a problem using a traditional algorithm, only to think later to yourself, Oh, I could have done that so much more simply! Guiding students through the six Fluency Actions as they solve problems, there is (or should be) a voice in their head asking and saying things like, "Is there a shorter method? This seems to be going nowhere," and "Does this answer make sense?" Fluency includes checks for reasonableness throughout the process

FIGURE 1.5 ● Procedural Fluency Components, Actions, and Checks for Reasonableness



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of solving the problem. Figure 1.5 layers reasonableness as part of the comprehensive description of procedural fluency.

Put more simply and in language that is student-friendly, here are the three opportunities to check for reasonableness:

Choose. Choose a strategy that is efficient based on the numbers in the problem.

Change. Change the strategy if it is proving to be overly complex or unsuccessful.

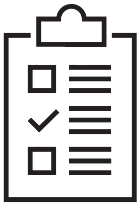
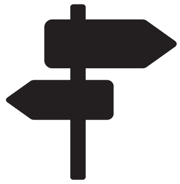
Check. Check to make sure the result makes sense.

TEACHING TAKEAWAY

Explicitly teaching the Choose, Change, Check metacognitive process for checking reasonableness will help students develop fluency and confidence in themselves.

These are quick actions that frame a metacognitive conversation but are rarely explicitly taught or recognized. Teaching and reinforcing these reasonableness checks with your students will greatly aid in their fluency. Explicitly teaching the Choose, Change, Check metacognitive process for checking reasonableness can help students develop fluency and confidence in themselves. One way to explicitly attend to reasonableness is to provide students with Question Cards (see Figure 1.6). Students have these cards for reference as they think through problems individually, with a partner, or a small group.

FIGURE 1.6 ● Choose, Change, Check Reflection Card for Students

CHECKS FOR REASONABLENESS		
Choose 	Change 	Check 
Is this something I can do in my head? What strategy makes sense for these numbers?	Is my strategy going well, or should I try a different approach? Does my answer so far seem reasonable?	Is my answer close to what I anticipated it might be? How might I check my answer?

Icon sources: Choose by iStock.com/Enis Aksoy; Change by iStock.com/Sigit Mulyo Utomo; Check by iStock.com/Indigo Diamond

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In the Common Core State Standards (CCSS) Mathematical Practices (MP), reasonableness is addressed in both MP1—*Make sense and persevere*—and MP8—*Look for and express regularity in repeated reasoning* (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010):

> MP1: Mathematically proficient students ... plan a solution pathway rather than simply jumping into a solution attempt ... monitor and evaluate their progress and change course if necessary ... [and] check their answers to problems ... continually ask[ing] themselves, "Does this make sense?"

> MP8: Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts.... As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Reasonableness is certainly underemphasized in standards documents. In the CCSS, beyond the mention in MP8, reasonableness is mentioned in only one standard at Grades 3, 4, 5, and 7. The other grades have no mention of it. Yet the questions like "Is there a shorter method?" and "Does this answer make sense?" are clearly essential to doing mathematics. Infusing reasonableness into the curriculum is largely the responsibility of the teachers and leaders who design lessons, units, and curriculum. Routines are effective for reinforcing such underemphasized skills. Activity 1.3 contains one idea to add to your routine repertoire to help students practice checking for reasonableness.

ACTIVITY 1.3
ROUTINE: "IS IT REASONABLE?"

Materials: Three "________ is about ________" statements (see examples in the following chart)

Directions: Pose the first statement. Give students a cue for *Reasonable* and *Not Reasonable*.

For example, you might use sign language, with  for reasonable and  for not reasonable. Prompt students to make a private decision and wait for a "Show Me" request. Students share their decision and discuss why. Alternatively, small groups can discuss which are reasonable or not and then share with whole class.

REASONABILITY STATEMENTS		
Subtraction Within 1,000	**Multiplication With Decimals**	**Percentage**
985 – 328 is about 600	2.56 × 4 is about 10	$\frac{13}{40}$ is about 33%
549 – 98 is about 300	13.44 × 2.88 is about 26	$\frac{11}{24}$ is about 33%
671 – 443 is about 300	4.75 × 5 is about 25	$\frac{17}{30}$ is about 60%

Source: signs for R and N by iStock.com/Jayesh

The "Is It Reasonable" routine in Activity 1.3 helps students develop ways to check for reasonableness. But if reasonableness is limited to a routine, students won't develop the metacognitive practice of thinking "is this reasonable" as they are solving problems embedded in their classroom tasks or homework. As indicated in Figure 1.5, three of the six Fluency Actions include attending to reasonableness. Importantly, these reasonableness Fluency Actions occur before you start solving a problem, during the solving, and at the conclusion. Developing procedural fluency, then, includes helping students develop the metacognitive practices throughout solving a problem. One way to do this is to model "Ask-Yourself Questions" (see Figure 1.7). Ask-Yourself Questions are initially modeled by the teacher to make such thinking visible to students, with the intent that students will internalize the questions as they solve problems independently (Kelemanik et al., 2016).

Stop & Reflect

How might you infuse these Ask-Yourself Questions into your classroom or school?

FIGURE 1.7 ● Reasonableness Ask-Yourself Questions

CHOOSE

Related Fluency Action: Selects an appropriate strategy

Before you solve, ask yourself these questions:

- *Is this something I can do in my head?*
- *Is the strategy or method I am considering a reasonable approach for the numbers in this problem?*
- *Is it reasonable to use the standard algorithm for this problem (or is there a shorter method)?*
- *What is a good estimate for the answer?*

CHANGE

Related Fluency Action: Trades out or adapts strategy

During the solving, ask yourself these questions:

- *Is this answer I got [partway through a process] reasonable?*
- *Is this amount of 'mess' reasonable, or did I make a mistake or pick a bad method?*
- *Should I trade out my strategy?*
- *How might I adapt my strategy?*

CHECK

Related Fluency Action: Gets correct answer

After solving, ask yourself these questions:

- *Is this answer close to what I anticipated it might be?*
- *Does my answer make sense?*
- *How can I check to see if my answer is correct?*

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Did you notice that in the CCSS Mathematical Practices and in the Ask-Yourself Questions, "reasonable" occurs in three phases of solving a problem (not just at the end)? Reasonableness throughout the problem-solving process must be modeled and discussed frequently. You can use the Ask-Yourself Questions to craft anchor charts to support students as they develop aspects of reasonableness. That is not to suggest that you stop students at three points to ask them to check for reasonableness, but rather to have reasonableness embedded in the process of solving a problem. If they only check their answer at the end, it can be too late—they may have spent an unnecessary amount of time with an approach that was not a good method in the first place.

TEACHING TAKEAWAY

Put Ask-Yourself Questions on anchor charts to support students as they develop aspects of reasonableness.

So when students choose a strategy, we want them to think about that strategy being a good fit or a good idea for solving that specific problem. Is the strategy reasonably useful or efficient? This takes practice. Activity 1.4 uses worked examples to focus on *choosing*. Students decide and discuss if the selected strategy was a good choice or not. The activity lends to journaling, independent work, or even homework and can be the focus of a rich classroom discussion that supports fluency and reasonableness (as there may be disagreements).

ACTIVITY 1.4
FOCUS TASK: GOOD CHOICE OR BAD CHOICE

Materials: Set of problem(s), each with a strategy to critique (see examples in Figure 1.8)

Directions: Pose one problem, along with a strategy explanation. Give students a minute to decide if the strategy is a good choice or bad choice. First, ask students to explain what the student did. Second, have students tell if they think the strategy was a good choice or bad choice and why. Third, ask students to offer alternatives for the problems where they decide the example is a bad choice. *Note*: This is an efficiency discussion with students, and in the end, the purpose is to attend to choice, not to agree on an absolute answer.

FIGURE 1.8 ● Examples of Problems for Good Choice or Bad Choice Activity

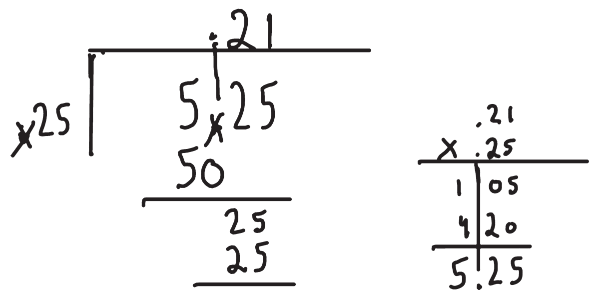
EXAMPLES	STRATEGY	GOOD CHOICE OR A BAD CHOICE? WHY?
37 + 74	Jimmy counted up, by ones, from 74.	
4,260 ÷ 60	Zoe broke 4,260 apart into 4,200 + 60 and divided both parts by 60 and then added the answers back together.	
$\frac{3}{8}+2\frac{4}{16}$	Brendan converted $2\frac{4}{16}$ to $\frac{36}{16}$, changed $\frac{3}{8}$ to $\frac{6}{16}$, and added to get $\frac{42}{16}$. Then, he changed $\frac{42}{16}$ to a mixed number.	

This activity grows into comparing two worked examples, which is highly effective in supporting student development of flexibility, a key Fluency Action (Rittle-Johnson et al., 2009).

While reasonableness is something to attend to more intentionally, there are some practices to avoid:

1. Don't ask about reasonableness without first building a concept of what reasonable means.
2. Don't treat reasonableness as another step of a procedure.

3. Don't try to use a "standard algorithm" for checking for reasonableness. In other words, applying the inverse operation to a problem may check one's accuracy with a procedure but does not necessarily determine if the results are reasonable. For example, see this student's thinking: The original problem was 5.25 ÷ .25. She divided and moved the decimal point. To check her work, she multiplied and moved the decimal down. In both cases, her work seems related to lining up decimals for addition and subtraction.



FLUENCY FOR EACH AND EVERY STUDENT

Fluency is an access issue. Think about it. Determining an efficient strategy, moving between strategies, and determining if a solution is reasonable does not happen if access to substantial instruction and meaningful practice of appropriate strategies is not provided. Consider 398 + 245 from earlier in the chapter. The strategies of Compensation or Make Hundreds (Make Tens extended) are not going to be in a student's repertoire if instruction has focused solely on the standard algorithm.

Stop & Reflect

How might these questions be used to pursue equitable mathematics teaching?

1. If only one process is taught, when do students learn about different strategies for becoming flexible and efficient?
2. When do students have an opportunity to make their own meaning and realize their own fluency?
3. What inequities manifest when some students have access to robust fluency instruction while others do not?
4. What toll does this lack of opportunity to learn take on individual students' mathematics identities?

MATHEMATICS IDENTITY AND AGENCY THROUGH FLUENCY

A mathematics identity is a deeply held belief students hold about themselves as mathematicians (Aguirre et al., 2013). Mathematical identities include students' sense of competence as it relates to knowing and doing mathematics, as well as their vulnerability and/or confidence. Identities are shaped by experiences and interactions. We believe that fluency plays a significant role in shaping a student's mathematics identity.

A traditional approach to procedural fluency has been to introduce and practice specific steps and rules. Memorizing procedural rules without understanding leads to difficulty remembering and applying those procedures. A rush to the standard algorithm and memorizing procedures undermines students' confidence and may cause math anxiety, which negatively impacts student achievement (Boaler, 2015b; Jameson, 2014; Ramirez et al., 2018). Moreover, "learning" someone else's rules and unsuccessfully carrying them out leads to self-doubt and defeat. There is little question about the long-term negative effect of a use-this-method-here approach on one's mathematics identity.

Conversely, well-implemented fluency instruction, attending to all six Fluency Actions, can powerfully counter potential negative dispositions about math and doing math. Methods like Compensation and Make Tens (or Hundreds) are based on place value understandings, properties, and number relations (e.g., 398 is just 2 away from 400). These methods require conceptual understanding *and* support the development of conceptual understanding. Conceptual understanding establishes the logic and the "how." Understanding number relationships is empowering. You see you have choices and shortcuts. You understand the answer shouldn't be larger when you are multiplying a whole number by a fraction less than 1.

Effective teaching ensures that each child understands how to *use* relevant strategies and also has ample opportunities to *choose* among strategies. Learning different strategies opens the door to procedural fluency; learning to choose among those methods allows students to pass through that door. Such meaningful practice instills confidence and grows competence as students continue to become more and more proficient at using and choosing strategies and executing them efficiently and accurately. When students understand what they are doing, they identify as someone who can do math. Logically, students who understand what they are doing, why they are doing it, and can monitor their reasonableness are much more likely to have a positive disposition about math than students who are memorizing procedures and hoping they are correct. Together, conceptual understanding and procedural fluency spark belief in oneself and contribute to a positive mathematics identity. In short, fluent students *can* see themselves as "math people." They have agency.

TEACHING TAKEAWAY

Effective teaching ensures that each child understands how to *use* relevant strategies and also has ample opportunities to *choose* among strategies.

When people are able to participate and perform effectively in mathematics contexts, they have mathematical agency (Aguirre et al., 2013). It is the behavioral side of people's mathematics identity, or their identity-in-action. Students with mathematical agency see themselves as mathematical thinkers who understand what they are doing and feel they *can* solve a problem without being shown how. In fact, such students exhibit mathematical proficiency as defined by the National Research Council—exhibiting both conceptual understanding and procedural fluency, but also strategic competence, adaptive reasoning, and a productive disposition.

Because effective instruction of (real) fluency values actions—such as selecting, understanding, and evaluating strategies, as well as flexibility and reasonableness—students are able to develop strategic competence and adaptive reasoning. These competencies positively shape their mathematics identity, while also nurturing their mathematical agency. The mathematical agency of students is actualized when they choose a strategy to successfully solve a problem. Conversely, a sense of agency cannot develop when students are not given the opportunity to select a strategy or learn to check for reasonableness. And yet this is the culture of many classrooms: Students are shown or told how to solve problems and not encouraged to employ their own reasoning or strategy.

Tragically, many educators have the unproductive belief that students with disabilities or students who struggle require this type of support (i.e., being shown a step-by-step process with little or no attention to conceptual understanding or reasoning). They also believe that having students memorize just one method is in their best interest. This belief is a mistake for several reasons. First, memorizing is a weak learning strategy, particularly for students with disabilities. Instead, procedural fluency for students with disabilities should infuse research-based strategies such as using a concrete–semi-concrete–abstract (CSA) approach to learning procedures, think-alouds, peer-assisted learning, and explicit strategy instruction (Gersten et al., 2009; NCTM, 2007). Second, memorizing without understanding can lead to a negative mathematics identity ("I don't understand this") and no sense of mathematical agency ("I don't know how to find an answer"). Students with disabilities benefit every bit as much as other students from an instructional focus on fluency with efficiency, flexibility, and accuracy. Every student deserves opportunities to develop procedural fluency and thereby also develop positive mathematical identities and agency.

TEACHING TAKEAWAY

Students with disabilities benefit as much as other students from an instructional focus on fluency with efficiency, flexibility, and accuracy.

Stop & Reflect

What do you see as the relationships among identity, agency, and procedural fluency?

PRODUCTIVE BELIEFS ABOUT ACCESS AND EQUITY FOR FLUENCY

NCTM's *Principles to Actions: Ensuring Mathematics Success for All* (2014) poses that effective teaching practices and student success are only possible when essential elements of mathematics programs are in place. First and foremost is the element of commitment to access and equity. That element is defined and described through productive beliefs about access and equity in mathematics. For example, an unproductive belief related to access and equity is that "only high-achieving or gifted students can reason about, make sense of, and persevere in solving challenging mathematics problems" (NCTM, 2014, p. 64) versus a related productive belief:

> All students are capable of making sense of and persevering in solving challenging mathematics problems and should be expected to do so. Many more students, regardless of gender, ethnicity, and socioeconomic status, need to be given the support, confidence, and opportunities to reach much higher levels of mathematical success and interest. (NCTM, 2014, p. 64)

We see fluency instruction directly connected to each of those productive beliefs. We use those beliefs to frame access and equity for fluency. These beliefs shape what it means to provide equitable fluency instruction. Figure 1.9 offers five productive beliefs about the teaching and learning of procedural fluency.

FIGURE 1.9 ● Productive Beliefs About Procedural Fluency

1.	Procedural fluency is an attainable goal for each and every student. Each student is capable of developing a repertoire of strategies and learning skills at applying those strategies flexibly, efficiently, and accurately.
2.	Procedural fluency is a function of opportunity, experience, and effort. Differentiated supports enable each and every student to understand and use a range of strategies.
3.	Procedural fluency instruction is higher-order thinking, as students create strategies, generalize when to use a strategy, and explain why a strategy works. This increased level of thinking leads to greater understanding and performance for every student.
4.	Every student must have access to instruction and resources that attend to all procedural fluency components and actions.
5.	Having a range of ideas and strategies for solving procedures enriches everyone's learning. Therefore, every student benefits from heterogenous grouping; conversely, homogeneous grouping (ability grouping) is detrimental to developing procedural fluency.

These beliefs are nonnegotiable and must be addressed at the forefront of implementing a schoolwide fluency plan. In essence, they provide the fertile ground from which equitable, effective teaching practices for procedural fluency can grow and develop.

EFFECTIVE TEACHING PRACTICES FOR FLUENCY INSTRUCTION

NCTM's Effective Mathematics Teaching Practices (NCTM, 2014, p. 10) are research-based, effective instructional practices that frame equitable, effective practice, ensuring every student has opportunity and access to a high-quality mathematics program. These practices must be directly connected to content across the curriculum, especially fluency topics. Figure 1.10 describes how each Effective Teaching Practice applies to fluency instruction.

FIGURE 1.10 ● NCTM's (2014) Effective Teaching Practices Connected to Fluency Instruction

TEACHING PRACTICE	APPLICATION TO FLUENCY INSTRUCTION
Establish mathematics goals to focus learning.	Goals for fluency lessons attend to all three components of fluency (Chapter 1) and are part of balanced assessment practices (Chapter 7). Fluency instruction is based on the progression of strategies (Chapter 3).
Implement tasks that promote reasoning and problem-solving.	Fluency tasks include instructions for students to select and use different strategies, and implementation of these tasks includes reflection on when particular strategies make sense and when they do not, attending to reasonableness (Chapters 3, 4, and 6).
Use and connect mathematical representations.	Strategies are taught with mathematical representations so that students see the inherent mathematical relationships (Chapters 3 and 4).
Facilitate meaningful mathematical discourse.	Students have opportunities to discuss and explain strategy selection, efficiency, and reasonableness during instruction and practice (Chapters 3, 4, 5, and 6).
Pose purposeful questions.	Students are asked to explain strategy selection, flaws, and relationships (Chapters 3, 4, 5, 6, and 7).
Build procedural fluency from conceptual understanding.	Strategies are developed from understanding of concepts, and conversely, using strategies strengthens students' understanding (every chapter, but particularly Chapters 3, 4, and 5).
Support productive struggle in learning mathematics.	Students have time and support to grapple with learning strategies and determining when they should employ a strategy. They have processing time to develop their own ideas about and utility with strategies (Chapters 2, 3, 4, and 6).
Elicit and use evidence of student thinking.	Efficiency, flexibility, accuracy, and reasonableness—in particular, the six observable Fluency Actions—are assessed in a variety of ways, and the information is used to establish goals and differentiated support (Chapters 4 and 7).

We recognize that equitable mathematics teaching practices do not come about easily and that we have a long way to go, as we have too many students who robotically implement algorithms that don't make sense to them. The good news is that a focus on fluency can positively impact a number of teaching practices. For example, think about implementing Activity 1.2, the game of *Strategories*. The goal is solidly focused on flexibility and efficiency; thus, the task is a reasoning task. There are opportunities for posing purposeful questions and engaging in meaningful discourse. As students create problems and talk to peers about their choices, you can support their struggle and gather evidence. The key is that the goal and the task actually focus on real fluency. And that is the purpose of this book: to present what fluency goals mean (in this chapter), make visible what those strategies are and what quality fluency tasks and activities look like, and then ensure that we are also assessing fluency and communicating what fluency is to families and other stakeholders. The parenthetical notations in Figure 1.10 identify where this happens in this book.

FIGURING OUT FLUENCY: SETTING CLEAR GOALS

Procedural fluency is a critical, required component of balanced mathematics instruction. Approaches to fluency must shift—wherein students learn algorithms, but more importantly, learn *when* they need them … and when they don't. This decision-making, inherent in true fluency, is a critically important life skill. Beyond being a life skill, such decision-making (strategy selection) is important for higher-level mathematics and test-taking.

Most importantly, fluency is an equity issue. Students' mathematics identity and agency are shaped by the way in which we engage them in learning about diverse strategies and how we help them make decisions about using those strategies. When students are afforded the opportunity to make sense of procedures and select ones that make sense to them, they develop confidence and competence. When students explain a method to their peers and the teacher elevates that strategy as one for others to consider, students see themselves and their peers as doers of mathematics. Being fluent contributes to a productive disposition about mathematics, opens doors to a range of mathematics topics, and arms students with a skillset applicable to whatever they wish to pursue.

Figuring out fluency is not about finding magic bullets to fix students' fluency shortcomings. It is about understanding what fluency is and its importance to an equity agenda (discussed in this chapter). We believe that advancing our collective fluency work is a call to action! And this call to action requires acknowledging that teaching for procedural fluency is mired in myths and misconceptions. We call them Fluency Fallacies, and they are the focus of Chapter 2.

Talk About It

Figuring out fluency begins with knowing what fluency is and the role it plays in students' mathematics identities and agencies. These prompts are designed to help you reflect on this chapter, as well as consider ways to focus on fluency in your classroom and your setting:

1. How does fluency, as described in Chapter 1, compare with what you previously thought about fluency? How would you describe fluency to colleagues? To families?
2. Which of the three fluency components are most challenging for you as a doer of math? As a teacher? Why?
3. Which of the Fluency Actions get most of your instructional attention? Which is featured the least?
4. How would you respond to someone's declaration that fluency is not about equity?
5. What examples can you share of how a student's mathematics identity and/or mathematical agency has been impacted by lessons on procedural fluency?
6. Which beliefs about accessible and equitable fluency instruction are most prevalent in your team, school, or district?

Act On It

Figuring out fluency takes a community. With your colleagues, consider engaging in these productive activities to increase your focus on fluency:

1. **"You're in the Driver's Seat" activity**. This mixer is a great connect to the meaning of procedural fluency and will be effective with your faculty or can be adapted for students (Walters & Bachman, 2020). With a partner, tell each other turn-by-turn directions for how to get from campus to your home (or favorite restaurant or some other place you frequent). After a couple of minutes, switch partners, but this time tell a different way to get to your chosen destination. Repeat a third time (as time allows). Ask the group these questions:
   - a. Which route do you usually pick and why?
   - b. Why might you use an alternate route?
   - c. How did you come to know these routes?

2. **Deconstruct fluency-focused content standards**. Select a procedure-related standard and outline what the three components of fluency might look like for a student who has "mastered" that standard. Instead, or in addition, create a list of "look-fors" for each of the six Fluency Actions. Finally, consider what the expectations for reasonableness might look like.

3. **My Mathematics Identity project.** Do an "identity" activity project with your students, probing into what they think someone who is good at math "looks like" and asking which of those characteristics they think they have. Analyze these with an eye on fluency (***spoiler alert***: being fast and knowing facts may come up as "good at math," which is not actually what it means to be good at math!). Looking for some excellent activities? Both of these articles published in NCTM's journal ***Mathematics Teaching in the Middle School*** offer a collection of ideas and are easily adapted to K–5 students:

   a. "Developing Mathematics Identity" by Kasi Allen and Kemble Snell (March 2016)
   b. "Exploring Our Complex Math Identities" by Keith Leatham and Diane Hill (November 2010)

4. **Effective teaching of procedural fluency.** Select one of NCTM's Effective Teaching Practices (see Figure 1.10; 2014, p. 10) and identify ways to integrate working on that practice with improving fluency instruction. The first two are good places to start. For example, for the first practice listed, you might consider how to write objectives that reflect a fluency focus, as well as consider ways to revisit that lesson goal throughout the lesson.