What Your Colleagues Are Saying . . .

*"This book is a 'must-go-to' for classroom teachers as well as for coaches and leaders who support teachers in implementation of the Common Core Mathematics Standards. Connecting both the content standards and the practice standards with the effective teaching practices (*Principles to Actions, *NCTM, 2014), Linda Gojak and Ruth Harbin Miles provide the practical resources to enhance teachers' understanding of the mathematics in the standards in ways that will bring them alive for all students. This is a book that will be the cornerstone of our professional development."*

—Cathy Martin, Director of PreK–12 Mathematics
Denver Public Schools, Denver, CO

"The Common Core Companion series for ELA became a runaway bestseller because it made instructional planning to address the standards manageable—and meaningful. Jim Burke and his co-authors put best practices first, equipping teachers with the 'what-it-looks-like' teaching ideas for each and every standard. Teachers have been clamoring for a math version for more than a year, and now it's here! Linda Gojak and Ruth Harbin Miles do an amazing job demystifying the standards for grades 3–5. The authors have created a format and design that will have teachers dog-earing this book as they plan for instruction!"

—Leslie Blauman, Author of *The Common Core Companion, Grades 3–5*

"This companion supports implementation of the Common Core Mathematics Standards with attention to the instructional shifts: focus, coherence, and rigor. The standard-by-standard examples provide an image of what the standards looks like in the classroom for both teachers and students. The attention to students' conceptions helps teachers plan with student thinking in mind and contributes to the collaborative work of grade-level teams."

—Nicole Rigelman, Associate Professor
Portland State University, Portland, OR

"This practical, exemplary resource supports teachers in their understanding and learning of the Common Core Mathematics Standards. The book does an excellent job of unpacking what the standards mean and makes explicit connections both to the Standards for Mathematical Practice and common student misconceptions. It is an ideal choice for a book study at either the grade or school level."

—Jeffrey Shih, Associate Professor
University of Nevada, Las Vegas

The Common Core Companion at a Glance

Operations and Algebraic Thinking

Domain Overview

GRADE 3
The major work of this domain in Grade 3 is to develop students' conceptual understanding of multiplication and division by using concrete materials to model multiplication and then relate their understanding of multiplication to division. Multiplication problem situations provide a context for understanding multiplication as finding the total number of items given a number of equal groups and the number of items in each group. Division problem situations develop the meaning of division and how it is related to multiplication. When you know the total number of items and the number of groups, you can determine how many items in a group, or, when you know the total number of items and the number of items in a group, you can find the number of groups. All of these activities culminate in the expectation that students will demonstrate fluency with multiplication and division within 100 using single-digit factors.

GRADE 4
Students in Grade 4 continue to solve problems using the four operations with whole numbers. New to this grade level are problem situations that involve multiplicative comparisons. Students become familiar with factors and multiples and how they relate to prime and composite numbers. They work in a variety of contexts to generate and analyze patterns.

GRADE 5
In preparation for the Expressions and Equations domain in grades 6–8, fifth graders begin to explore, interpret, and evaluate numerical expressions. Work with patterns that began in Grade 4 extends to generating patterns, forming ordered pairs, graphing on a coordinate plane, and then analyzing the graphical representations.

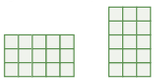
Domain Overview: Gives a brief description of the big ideas, allowing you to see how the mathematical ideas develop across grade levels.

Suggested Materials for This Domain: Provides teachers with a list of materials that will be helpful in introducing the concepts in this domain. "Reproducible" indicates that there is a handout that you can use to make multiple copies in the Resources section in the back of this book.

SUGGESTED MATERIALS FOR THIS DOMAIN

3	4	5	
✓	✓	✓	Hundreds chart (Reproducible 1)
✓	✓	✓	Chips, counters
✓	✓	✓	Cups, containers, other objects to represent "groups"
✓	✓	✓	Place value chart to hundreds (Reproducible 2)
✓	✓	✓	Square tiles
✓	✓	✓	Grid paper (Reproducible 3)
✓	✓	✓	Pattern blocks
✓	✓	✓	Number cards (such as a deck of playing cards)

KEY VOCABULARY

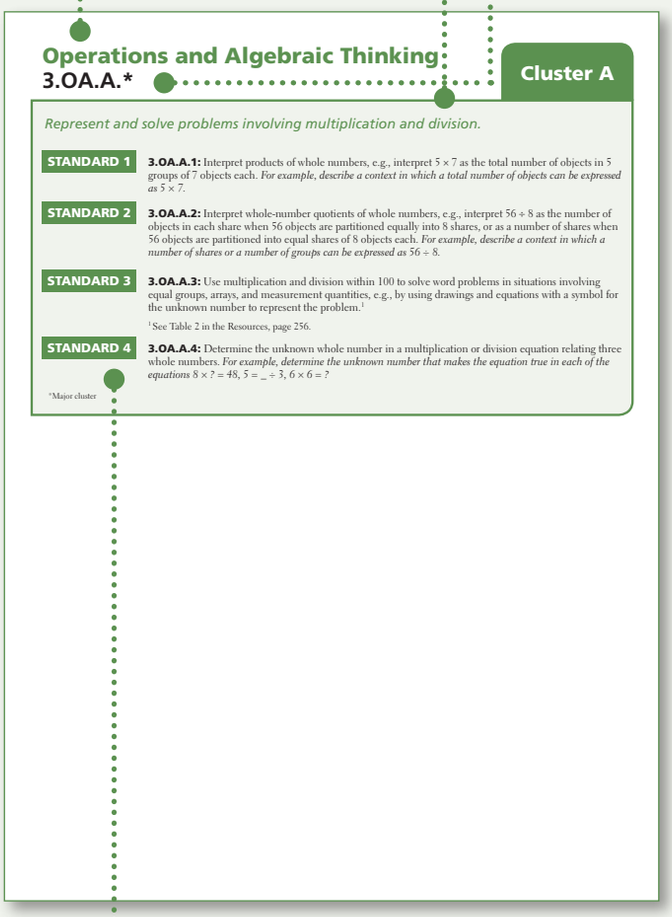
3	4	5	
✓	✓	✓	**add** to combine or join together related words: *add, and, plus, join, put together, (+)*
✓	✓	✓	**addend** any of the numbers added to find a sum
✓	✓	✓	**area model** a concrete model for multiplication or division made up of a rectangle. The length and width represent the factors and the area represents the product.  3 × 5 5 × 3
✓	✓	✓	**array model** a concrete model for multiplication in which items are arranged in rows and columns. Each row (or column) represents the number of groups and each column (or row) represents the number of items in a group.  5 × 4 = 20 5 rows of 4 = 20
✓	✓	✓	*** associative property of multiplication** an extension of the commutative property; to change the order and group two factors to find convenient products (such as 10) in order to make the multiplication easier. Students may begin to use parentheses at this level. 7 × 8 × 5 = 7 × (8 × 5) = 7 × 40 = 280
✓	✓	✓	*** commutative property of multiplication** reversing the order of the factors does not change the product 8 × 5 = 40 and 5 × 8 = 40 therefore the product of 8 × 5 = 5 × 8

Key Vocabulary: Vocabulary included in the domain with grade levels at which that term is used. This terminology can be used for building a word wall in the classroom. Students should be able to use these terms in talking about mathematics in discussions unless otherwise noted. Standard for Mathematical Practice 6: Attend to Precision calls for students to use mathematical terminology appropriately.

Domain: General mathematical topic for this group of Standards.

Cluster: Statements that summarize groups of related Standards. Note that Standards from different clusters may sometimes be closely related, because mathematics is a connected subject.

Identifying number for this cluster: Grade, domain, cluster



Operations and Algebraic Thinking
3.OA.A.*

Cluster A

Represent and solve problems involving multiplication and division.

STANDARD 1 **3.OA.A.1:** Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5 × 7.

STANDARD 2 **3.OA.A.2:** Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as* 56 ÷ 8.

STANDARD 3 **3.OA.A.3:** Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

[1] See Table 2 in the Resources, page 256.

STANDARD 4 **3.OA.A.4:** Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations* 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?

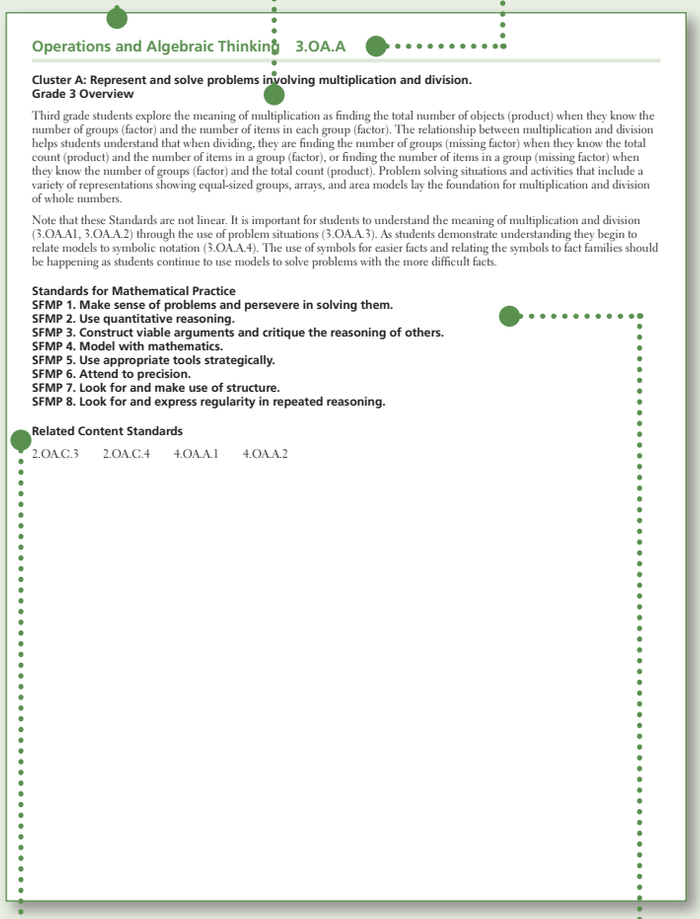
*Major cluster

Standards: Mathematical statements that define what students should understand and be able to do.

Grade, domain, cluster

Domain

Each cluster begins with a brief description of the mathematics in that cluster.



Operations and Algebraic Thinking 3.OA.A

Cluster A: Represent and solve problems involving multiplication and division.
Grade 3 Overview

Third grade students explore the meaning of multiplication as finding the total number of objects (product) when they know the number of groups (factor) and the number of items in each group (factor). The relationship between multiplication and division helps students understand that when dividing, they are finding the number of groups (missing factor) when they know the total count (product) and the number of items in a group (factor), or finding the number of items in a group (missing factor) when they know the number of groups (factor) and the total count (product). Problem solving situations and activities that include a variety of representations showing equal-sized groups, arrays, and area models lay the foundation for multiplication and division of whole numbers.

Note that these Standards are not linear. It is important for students to understand the meaning of multiplication and division (3.OA.A1, 3.OA.A.2) through the use of problem situations (3.OA.A.3). As students demonstrate understanding they begin to relate models to symbolic notation (3.OA.A.4). The use of symbols for easier facts and relating the symbols to fact families should be happening as students continue to use models to solve problems with the more difficult facts.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Related Content Standards
2.OA.C.3 2.OA.C.4 4.OA.A.1 4.OA.A.2

Related Content Standards: Provides a list of Standards connected to this topic in other grade levels as well as Standards in this grade level related to this topic that are in other domains. We recommend you look at the related Standards as you plan your instruction for this cluster.

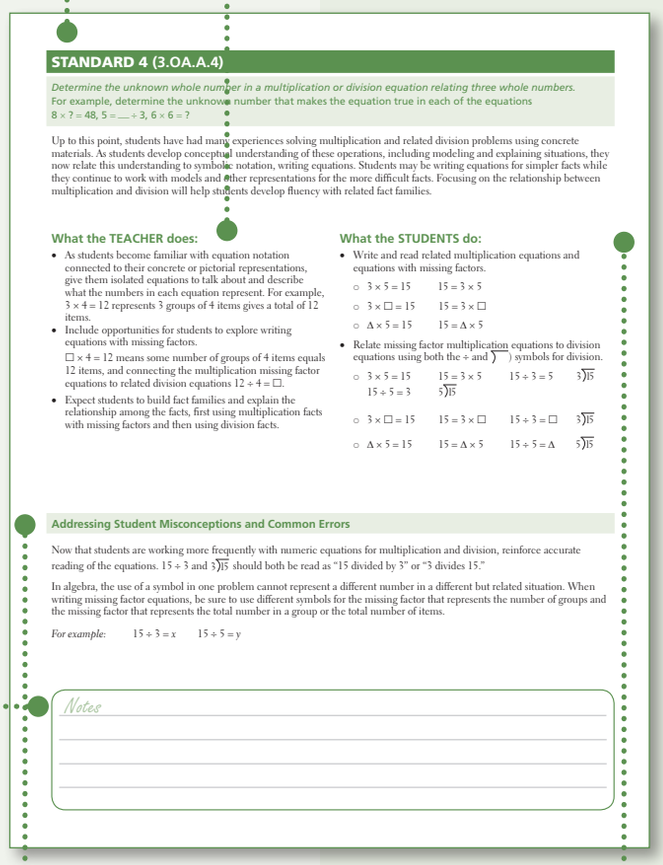
Standards for Mathematical Practice: Although it is likely you will use a variety of Standards for Mathematical Practice in teaching each cluster, this section gives examples of how you might incorporate some of the practices into your instruction on this topic.

You will find the following components for each Standard in the cluster:

Standard: The Standard as written in the Common Core followed by an explanation of the meaning of the mathematics in that Standard, including examples.

Notes: We have included space beneath each Standard for you to take notes as you study its mathematical content. This might include vocabulary, materials, resources you want to use, or an explanation of the Standard in your own words.

What the TEACHER does: An overview of actions the teacher might take in introducing and teaching the Standard. This is not meant to be all-inclusive, but rather to give you an idea of what classroom instruction might look like. We included illustrations of how to use materials to teach a concept when using models and representations is called for in the Standard.



STANDARD 4 (3.OA.A.4)

Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations $8 \times ? = 48$, $5 = \_\_ \div 3$, $6 \times 6 = ?$

Up to this point, students have had many experiences solving multiplication and related division problems using concrete materials. As students develop conceptual understanding of these operations, including modeling and explaining situations, they now relate this understanding to symbolic notation, writing equations. Students may be writing equations for simpler facts while they continue to work with models and other representations for the more difficult facts. Focusing on the relationship between multiplication and division will help students develop fluency with related fact families.

What the TEACHER does:

- As students become familiar with equation notation connected to their concrete or pictorial representations, give them isolated equations to talk about and describe what the numbers in each equation represent. For example, $3 \times 4 = 12$ represents 3 groups of 4 items gives a total of 12 items.
- Include opportunities for students to explore writing equations with missing factors.
  $\square \times 4 = 12$ means some number of groups of 4 items equals 12 items, and connecting the multiplication missing factor equations to related division equations $12 \div 4 = \square$.
- Expect students to build fact families and explain the relationship among the facts, first using multiplication facts with missing factors and then using division facts.

What the STUDENTS do:

- Write and read related multiplication equations and equations with missing factors.
  - $3 \times 5 = 15$ $\quad 15 = 3 \times 5$
  - $3 \times \square = 15$ $\quad 15 = 3 \times \square$
  - $\Delta \times 5 = 15$ $\quad 15 = \Delta \times 5$
- Relate missing factor multiplication equations to division equations using both the ÷ and $)\overline{\quad}$ symbols for division.
  - $3 \times 5 = 15$ $\quad 15 = 3 \times 5$ $\quad 15 \div 3 = 5$ $\quad 3\overline{)15}$
    $15 \div 5 = 3$ $\quad 5\overline{)15}$
  - $3 \times \square = 15$ $\quad 15 = 3 \times \square$ $\quad 15 \div 3 = \square$ $\quad 3\overline{)15}$
  - $\Delta \times 5 = 15$ $\quad 15 = \Delta \times 5$ $\quad 15 \div 5 = \Delta$ $\quad 5\overline{)15}$

Addressing Student Misconceptions and Common Errors

Now that students are working more frequently with numeric equations for multiplication and division, reinforce accurate reading of the equations. $15 \div 3$ and $3\overline{)15}$ should both be read as "15 divided by 3" or "3 divides 15."

In algebra, the use of a symbol in one problem cannot represent a different number in a different but related situation. When writing missing factor equations, be sure to use different symbols for the missing factor that represents the number of groups and the missing factor that represents the total number in a group or the total number of items.

For example: $15 \div 3 = x$ $\quad 15 \div 5 = y$

Notes

Addressing Student Misconceptions and Common Errors: Each Standard concludes with a description of possible student misconceptions or common student errors around the Standard and suggested actions to address those misconceptions or errors.

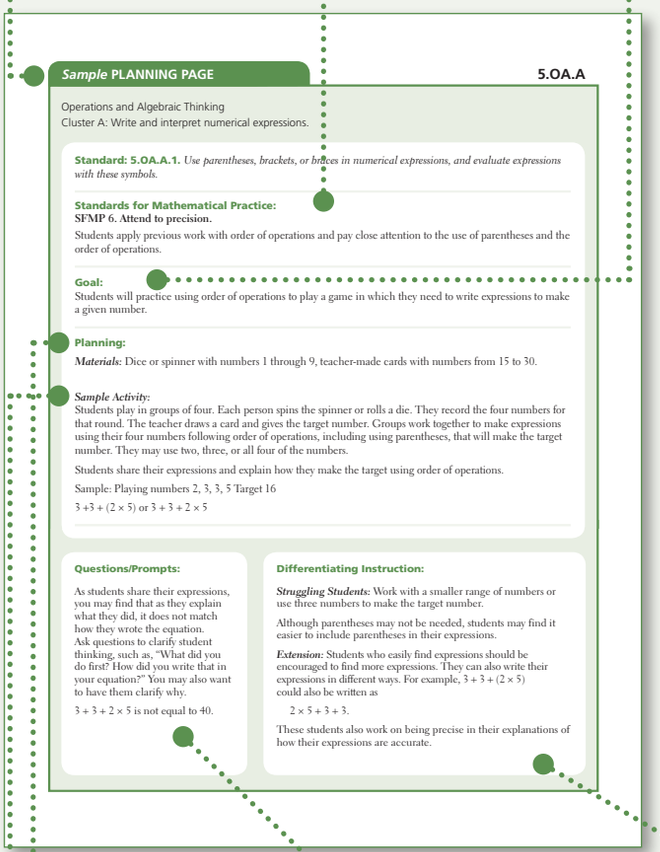
What the STUDENTS do: Some examples of what the students might be doing as they explore and begin to understand the Standard. Again, this is not intended to be directive but rather to frame what student actions might look like.

Sample Planning Page: We have provided one complete sample planning page for one Standard at the end of each cluster in each grade level. Although it may not be the final lesson plan, it does provide the areas you should consider while planning your lessons.

Goal: What is the purpose of this activity and how does it connect to previous (and future) ideas?

Standards for Mathematical Practice: What mathematical practices can you emphasize in this activity?

Planning Page: A planning template is provided at the end of each cluster. This template is provided for your use as you consider instructional actions around a particular Standard. You might want to make copies of this page and use them for each Standard within the cluster. This is not intended to be an all-inclusive lesson plan. Rather, it gives you a place to record your thoughts about teaching a mathematical topic as you read the Standard.



Sample PLANNING PAGE 5.OA.A

Operations and Algebraic Thinking
Cluster A: Write and interpret numerical expressions.

Standard: 5.OA.A.1. *Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.*

Standards for Mathematical Practice:
SFMP 6. Attend to precision.
Students apply previous work with order of operations and pay close attention to the use of parentheses and the order of operations.

Goal:
Students will practice using order of operations to play a game in which they need to write expressions to make a given number.

Planning:
Materials: Dice or spinner with numbers 1 through 9, teacher-made cards with numbers from 15 to 30.

Sample Activity:
Students play in groups of four. Each person spins the spinner or rolls a die. They record the four numbers for that round. The teacher draws a card and gives the target number. Groups work together to make expressions using their four numbers following order of operations, including using parentheses, that will make the target number. They may use two, three, or all four of the numbers.

Students share their expressions and explain how they make the target using order of operations.

Sample: Playing numbers 2, 3, 3, 5 Target 16

$3 + 3 + (2 \times 5)$ or $3 + 3 + 2 \times 5$

Questions/Prompts:

As students share their expressions, you may find that as they explain what they did, it does not match how they wrote the equation. Ask questions to clarify student thinking, such as, "What did you do first? How did you write that in your equation?" You may also want to have them clarify why.

$3 + 3 + 2 \times 5$ is not equal to 40.

Differentiating Instruction:

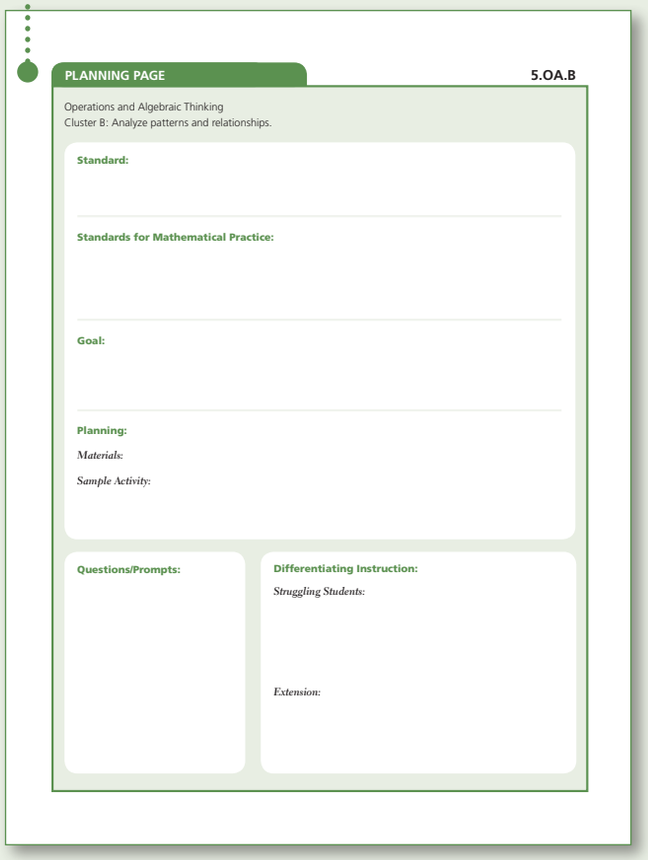
Struggling Students: Work with a smaller range of numbers or use three numbers to make the target number.

Although parentheses may not be needed, students may find it easier to include parentheses in their expressions.

Extension: Students who easily find expressions should be encouraged to find more expressions. They can also write their expressions in different ways. For example, $3 + 3 + (2 \times 5)$ could also be written as

$2 \times 5 + 3 + 3$.

These students also work on being precise in their explanations of how their expressions are accurate.



PLANNING PAGE 5.OA.B

Operations and Algebraic Thinking
Cluster B: Analyze patterns and relationships.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

Planning: What materials will you use to teach this Standard?

Sample Activity: An example of an activity that addresses this standard is provided.

Questions/Prompts: It is important to anticipate student thinking throughout the activity. Think about the questions or prompts you might give to help build student understanding and encourage student thinking so you do not find yourself telling students what to do.

Differentiating Instruction: How can you tweak the activity to address the needs of students who are struggling? How can you extend the activity for students who demonstrate understanding of the mathematics?

Resources: In the Resources section at the end of the book you will find an overview of each practice for teachers of grades 3–5 to consider and implement: Table 1, Addition and Subtraction Situations, Grades 3–5, which explains problem solving situations for addition and subtraction, and Table 2, Multiplication and Division Situations, Grades 3–5, which explains problem solving situations for multiplication and division and provides strategic competencies for students. Other resources include Table 3, which offers an overview of the Standards for Mathematical Practice and what each practice Standard means for students in grades 3–5; Table 4, the effective teaching practices from NCTM's *Principles to Actions;* and reproducibles for some of the materials recommended for each grade level.

An example of a problem that exemplifies the situation.

Equation(s) that represent the situation.

Table 1 Addition and Subtraction Situations, Grades 3–5

Situation	Problem	Equation(s)
Add to—result unknown	Frank had 235 pennies. Mark gave him 156 more. How many pennies does Frank have?	**$235 + 156 = p$**
Add to—change unknown	Frank had 235 pennies. Mark gave him some more. Now Frank has 391 pennies. How many pennies did Mark give to Frank?	**$235 + p = 391$**
Add—start unknown	Frank had some pennies in his piggy bank. Mark gave him 156 more. Now Frank has 391 pennies. How many pennies did Frank have at the beginning?	**$p + 156 = 391$**
Take from—result unknown	Frank had 45 pennies. He spent 29 pennies on a package of jawbreakers. How many pennies does he have left?	**$45 - 29 = p$**
Take from—change unknown	Frank had 45 pennies. He spent some pennies on a pack of jawbreakers. Now Frank has 16 pennies. How much did he spend on the jawbreaker?	**$45 - p = 16$**
Take from—start unknown	Frank had some pennies in his bank. He spent 29 pennies on a package of jawbreakers. Now he has 16 pennies. How many pennies did Frank have in his bank?	**$p - 29 = 16$**
Put together take apart—total unknown	Anna has been saving coins. She has 348 pennies and 267 nickels. How many coins does she have?	**$348 + 267 = p$**
Put together take apart—addend unknown	Anna 615 coins in her piggy bank. She has 348 pennies and the rest are nickels. How many coins are nickels?	**$615 = 348 + p$**
Put together take apart—addends unknown	Anna has 11 coins. Some are pennies and some are nickels. How many pennies and how many nickels could Anna have?	**$1 + 10 = 11$** **$2 + 9 = 11$** **$3 + 8 = 11$** **$4 + 7 = 11$** **$5 + 6 = 11$** **$6 + 5 = 11$** **$7 + 4 = 11$** **$8 + 3 = 11$** **$9 + 2 = 11$** **$10 + 1 = 11$**

Various problem situations for addition and subtraction.

CCSS Where to Focus Grade 4 Mathematics

CCSS WHERE TO FOCUS GRADE 4 MATHEMATICS

MATH (MATHEMATICS) 4 (GRADE 4) F (FOCUS)

This document shows where students and teachers should spend the large majority of their time in order to meet the expectations of the Standards.

Not all content in a given grade is emphasized equally in the Standards. Some clusters require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice.

To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade.

Students should spend the large majority[1] of their time on the major work of the grade (■). Supporting work (□) and, where appropriate, additional work (○) can engage students in the major work of the grade.[2,3]

MAJOR, SUPPORTING, AND ADDITIONAL CLUSTERS FOR GRADE 4

Emphases are given at the cluster level. Refer to the Common Core State Standards for Mathematics for the specific standards that fall within each cluster.

Key: ■ Major Clusters □ Supporting Clusters ○ Additional Clusters

4.OA.A	■	Use the four operations with whole numbers to solve problems.
4.OA.B	□	Gain familiarity with factors and multiples.
4.OA.C	○	Generate and analyze patterns.
4.NBT.A	■	Generalize place value understanding for multi-digit whole numbers.
4.NBT.B	■	Use place value understanding and properties of operations to perform multi-digit arithmetic.
4.NF.A	■	Extend understanding of fraction equivalence and ordering.
4.NF.B	■	Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
4.NF.C	■	Understand decimal notation for fractions, and compare decimal fractions.
4.MD.A	□	Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
4.MD.B	□	Represent and interpret data.
4.MD.C	○	Geometric measurement: understand concepts of angle and measure angles.
4.G.A	○	Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

HIGHLIGHTS OF MAJOR WORK IN GRADES K–8

K–2	Addition and subtraction – concepts, skills, and problem solving; place value
3–5	Multiplication and division of whole numbers and fractions – concepts, skills, and problem solving
6	Ratios and proportional relationships; early expressions and equations
7	Ratios and proportional relationships; arithmetic of rational numbers
8	Linear algebra and linear functions

REQUIRED FLUENCIES FOR GRADE 4

4.NBT.B.4	Add/subtract within 1,000,000

1 At least 65% and up to approximately 85% of class time, with Grades K–2 nearer the upper end of that range, should be devoted to the major work of the grade. For more information, see Criterion #1 of the K-8 Publishers' Criteria for the Common Core State Standards for Mathematics www.achievethecore.org/publisherscriteria.

2 Refer also to criterion #3 in the K–8 Publishers' Criteria for the Common Core State Standards for Mathematics www.achievethecore.org/publisherscriteria.

3 Note, the critical areas are a survey of what will be taught at each grade level; the major work is the subset of topics that deserve the large majority of instructional time during a given year to best prepare students for college and careers.

STUDENT ACHIEVEMENT PARTNERS Find additional resources at achievethecore.org

Source: http://achievethecore.org

CCSS Where to Focus Mathematics: The major content focus for each grade level is identified on the grade-level focus charts included in the Resources.

Reproducibles: A variety of reproducibles can be downloaded from the companion website at **resources.corwin.com/mathematicscompanion3-5** and used by students in the classroom when working with concrete materials.

Reproducible 1. Hundreds Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Quick Reference Guide

GRADE 3

Operations and Algebraic Thinking 3.OA

A. Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5×7.
2. Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as* $56 \div 8$.
3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]
4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations* $8 \times ? = 48$, $5 = \_ \div 3$, $6 \times 6 = ?$

B. Understand properties of multiplication and the relationship between multiplication and division.

5. Apply properties of operations as strategies to multiply and divide.[2] *Examples: If* $6 \times 4 = 24$ *is known, then* $4 \times 6 = 24$ *is also known. (Commutative property of multiplication.)* $3 \times 5 \times 2$ *can be found by* $3 \times 5 = 15$, *then* $15 \times 2 = 30$, *or by* $5 \times 2 = 10$, *then* $3 \times 10 = 30$. *(Associative property of multiplication.) Knowing that* $8 \times 5 = 40$ *and* $8 \times 2 = 16$, *one can find* 8×7 *as* $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. *(Distributive property.)*
6. Understand division as an unknown-factor problem. *For example, find* $32 \div 8$ *by finding the number that makes* 32 *when multiplied by* 8.

C. Multiply and divide within 100.

7. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

D. Solve problems involving the four operations, and identify and explain patterns in arithmetic.

8. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]
9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

1. See Table 2 in the Resources, page 256.
2. Students need not use formal terms for these properties.
3. This Standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

Number and Operations in Base Ten 3.NBT

A. Use place value understanding and properties of operations to perform multi-digit arithmetic.[1]

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

1. A range of algorithms may be used.

Number and Operations—Fractions[1] 3.NF

A. Develop understanding of fractions as numbers.

1. Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$.
2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
   a. Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each

Note: More detail and examples from individual Standards can be found in the complete Standards document available at www.corestandards.org.

part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line.

b. Represent a fraction $\frac{a}{b}$ on a number line diagram by marking off a lengths $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line.

3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b. Recognize and generate simple equivalent fractions, e.g., $\frac{1}{2} = \frac{2}{4}$, $\frac{4}{6} = \frac{2}{3}$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form* $3 = \frac{3}{1}$; *recognize that* $\frac{6}{1} = 6$; *locate* $\frac{4}{4}$ *and 1 at the same point of a number line diagram.*

d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

1. Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

Measurement and Data 3.MD

A. Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[1] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[2]

B. Represent and interpret data.

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*

4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

C. Geometric measurement: Understand concepts of area and relate area to multiplication and to addition.

5. Recognize area as an attribute of plane figures and understand concepts of area measurement.

a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

7. Relate area to the operations of multiplication and addition.

a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

D. Geometric measurement: Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

8. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

1. Excludes compound units such as cm^3 and finding the geometric volume of a container.

2. Excludes multiplicative comparison problems (problems involving notions of "times as much"; see Table 2 in the Resources, page 256).

Geometry 3.G

A. Reason with shapes and their attributes.

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as* $\frac{1}{4}$ *of the area of the shape.*

GRADE 4

Operations and Algebraic Thinking 4.OA

A. Use the four operations with whole numbers to solve problems.

1. Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[1]
3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

B. Gain familiarity with factors and multiples.

4. Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

C. Generate and analyze patterns.

5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

1. See Table 2 in the Resources, page 256.

Number and Operations in Base Ten[1] 4.NBT

A. Generalize place value understanding for multi-digit whole numbers.

1. Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that* $700 \div 70 = 10$ *by applying concepts of place value and division.*
2. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
3. Use place value understanding to round multi-digit whole numbers to any place.

B. Use place value understanding and properties of operations to perform multi-digit arithmetic.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.
5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

1. Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

Number and Operations—Fractions[1] 4.NF

A. Extend understanding of fraction equivalence and ordering.

1. Explain why a fraction $\frac{a}{b}$ is equivalent to a fraction $\frac{(n \times a)}{(n \times b)}$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $\frac{1}{2}$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

B. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

3. Understand a fraction $\frac{a}{b}$ with $a > 1$ as a sum of fractions $\frac{1}{b}$.
   a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.
   b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples:* $\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$; $\frac{3}{8} = \frac{1}{8} + \frac{2}{8}$; $2\frac{1}{8} = 1 + 1 + \frac{1}{8} = \frac{8}{8} + \frac{8}{8} + \frac{1}{8}$.
   c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.
   d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.
4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.
   a. Understand a fraction $\frac{a}{b}$ as a multiple of $\frac{1}{b}$. *For example, use a visual fraction model to represent* $\frac{5}{4}$ *as the product* $5 \times \frac{1}{4}$, *recording the conclusion by the equation* $\frac{5}{4} = 5 \times \frac{1}{4}$.
   b. Understand a multiple of $\frac{a}{b}$ as a multiple of $\frac{1}{b}$, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express* $3 \times \frac{2}{5}$ *as* $6 \times \frac{1}{5}$, *recognizing this product as* $\frac{6}{5}$. *(In general,* $n \times \frac{a}{b} = \frac{(n \times a)}{b}$.*)*
   c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat* $\frac{3}{8}$ *of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*

C. Understand decimal notation for fractions, and compare decimal fractions.

5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[2] *For example, express* $\frac{3}{10}$ *as* $\frac{30}{100}$, *and add* $\frac{3}{10} + \frac{4}{100} = \frac{34}{100}$.
6. Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as* $\frac{62}{100}$; *describe a length as 0.62 meters; locate 0.62 on a number line diagram.*
7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

1. Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.
2. Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

Measurement and Data 4.MD

A. Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), . . .*
2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.
3. Apply the area and perimeter formulas for rectangles in real world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

B. Represent and interpret data.

4. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

C. Geometric measurement: Understand concepts of angle and measure angles.

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and can be used to measure angles.

   b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry 4.G

A. Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

GRADE 5

Operations and Algebraic Thinking 5.OA

A. Write and interpret numerical expressions.

1. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as* $2 \times (8 + 7)$. *Recognize that* $3 \times (18932 + 921)$ *is three times as large as* $18932 + 921$, *without having to calculate the indicated sum or product.*

B. Analyze patterns and relationships.

3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Number and Operations in Base Ten 5.NBT

A. Understand the place value system.

1. Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and $\frac{1}{10}$ of what it represents in the place to its left.

2. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

3. Read, write, and compare decimals to thousandths.

   a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times \frac{1}{10} + 9 \times \frac{1}{100} + 2 \times \frac{1}{1000}$.

   b. Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

4. Use place value understanding to round decimals to any place.

B. Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers using the standard algorithm.

6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Number and Operations—Fractions 5.NF

A. Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example,* $\frac{2}{3} + \frac{5}{4} = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}$. *(In general,* $\frac{a}{b} + \frac{c}{d} = \frac{(ad+bc)}{(bd)}$.*)*
2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result* $\frac{2}{5} + \frac{1}{2} = \frac{3}{7}$*, by observing that* $\frac{3}{7} < \frac{1}{2}$.

B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator ($\frac{a}{b} = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret* $\frac{3}{4}$ *as the result of dividing 3 by 4, noting that* $\frac{3}{4}$ *multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size* $\frac{3}{4}$*. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*
4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
   a. Interpret the product $\frac{a}{b} \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. *For example, use a visual fraction model to show* $\frac{2}{3} \times 4 = \frac{8}{3}$*, and create a story context for this equation. Do the same with* $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$. *(In general,* $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.*)*
   b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.
5. Interpret multiplication as scaling (resizing), by:
   a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.
   b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $\frac{a}{b} = \frac{(n \times a)}{(n \times b)}$ to the effect of multiplying $\frac{a}{b}$ by 1.
6. Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.
7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[1]
   a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for* $\frac{1}{3} \div 4$*, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that* $\frac{1}{3} \div 4 = \frac{1}{12}$ *because* $\frac{1}{12} \times 4 = \frac{1}{3}$.
   b. Interpret division of a whole number by a unit fraction, and compute such quotients. *For example, create a story context for* $4 \div \frac{1}{5}$*, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that* $4 \div \frac{1}{5} = 20$ *because* $20 \times \frac{1}{5} = 4$.
   c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, how much chocolate will each person get if 3 people share* $\frac{1}{2}$ *lb of chocolate equally? How many* $\frac{1}{3}$*-cup servings are in 2 cups of raisins?*

1. Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

Measurement and Data 5.MD

A. Convert like measurement units within a given measurement system.

1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

B. Represent and interpret data.

2. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

C. Geometric measurement: Understand concepts of volume and relate volume to multiplication and to addition.

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.
   a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.
   b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.
4. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.
5. Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.
   a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
   b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.
   c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Geometry 5.G

A. Graph points on the coordinate plane to solve real-world and mathematical problems.

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).
2. Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

B. Classify two-dimensional figures into categories based on their properties.

3. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
4. Classify two-dimensional figures in a hierarchy based on properties.

Standards for Mathematical Practice (3–5)

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

For a downloadable version of this Quick Reference Guide, visit the companion website at **resources.corwin.com/mathematicscompanion3-5**.

The Common Core Mathematics Companion: The Standards Decoded, Grades 3–5

What They Say, What They Mean, How to Teach Them

Linda M. Gojak

Ruth Harbin Miles

Series Creator: Jim Burke

Name: _____

Department: _____

Learning Team: _____

A JOINT PUBLICATION OF

CORWIN MATHEMATICS NCTM

CORWIN
A SAGE Company

FOR INFORMATION:

Corwin
A SAGE Company
2455 Teller Road
Thousand Oaks, California 91320
(800) 233-9936
www.corwin.com

SAGE Publications Ltd.
1 Oliver's Yard
55 City Road
London EC1Y 1SP
United Kingdom

SAGE Publications India Pvt. Ltd.
B 1/I 1 Mohan Cooperative Industrial Area
Mathura Road, New Delhi 110 044
India

SAGE Publications Asia-Pacific Pte. Ltd.
3 Church Street
#10-04 Samsung Hub
Singapore 049483

Printed in the United States of America

ISBN: 978-1-4833-8160-2

Series Creator: Jim Burke
Acquisitions Editor: Erin Null
Senior Associate Editor: Desirée A. Bartlett
Editorial Assistant: Andrew Olson
Production Editor: Melanie Birdsall
Copy Editor: Pam Suwinksy
Typesetter: C&M Digitals (P) Ltd.
Proofreader: Theresa Kay
Cover and Interior Designer: Scott Van Atta
Director of Marketing Strategy: Maura Sullivan

This book is printed on acid-free paper.

15 16 17 18 19 10 9 8 7 6 5 4 3 2 1

Contents

Part 4. Measurement and Data

Grade 3

Grade 4

Grade 5

Part 5. Geometry

Grade 3

For downloadable versions of the Quick Reference Guide, Reproducibles, and a Planning Page Template, visit the companion website at **resources.corwin.com/mathematicscompanion3-5**.

Acknowledgments

Thank you to all who have influenced my work as a K–8 mathematics teacher: the vision of Dr. James Heddens, my graduate advisor; Dr. Johnny Hill, who always pushed my thinking; Kay Gilliland, my mentor and friend; the many colleagues with whom I have worked and learned throughout my teaching career. Finally, thank you to all of my students who, through their work and questions, helped me to think more deeply about my own understanding of mathematics and to realize how lucky I am to have spent my time doing something I love!

—Linda M. Gojak

A very special thanks is due to the best teacher I have ever known, my incredible father, Dr. Calvin E. Harbin, who taught me to value my education and at the age of 99 is still modeling lifelong learning. Acknowledgment and thanks must also be given to my extraordinary mentors, Dr. Ramona Anshutz and Dr. Shirley A. Hill, who both inspired me to become a mathematics education leader. Their influence and guidance completely changed my life's work. Words could never express the thanks and credit I owe to my dear colleagues, Dr. Ted H. Hull and Dr. Don S. Balka, who are simply the best partners and team I have had the privilege to work with. Most important, I thank my loving husband, Sam Miles, for *always* being there for me.

—Ruth Harbin Miles

Letter to Grades 3–5 Teachers

Dear Teachers of Grades 3–5,

The Common Core Mathematics Companion: The Standards Decoded, Grades 3–5: What They Say, What They Mean, How to Teach Them is designed to support you as you help your students learn the mathematics they need to know and be able to do. This book includes critical mathematical ideas for each grade and is intended to be your guide to both the Content Standards and the Mathematical Practices. A brief overview for each Standard, along with effective teaching practices, mathematics vocabulary, suggested models, manipulatives, representations, and ideas for each Standard are included. The book is intended to help you make sense of the Content Standards and Mathematical Practices.

The Common Core State Standards for Mathematics (CCSSM) were developed to promote student achievement and have the potential for changing traditional classroom instruction across the United States. This is significant, because the Content Standards will help ensure students deeply understand the mathematics they are expected to learn. The Content Standards lay a foundation for the development of a rigorous, relevant, and coherent mathematics curriculum for every student and will help ensure all students are ready for their futures, including college and the workforce.

The CCSSM promote conceptual understanding and reasoning as well as skill proficiency. Included in the Common Core Mathematics document are five domains, clusters, and standards. The domains for 3–5 mathematics include the topics of Operations and Algebraic Thinking, Number and Operations in Base Ten, Number and Operations—Fractions, Measurement and Data, and Geometry. The Standards under each domain include developing conceptual understanding, skills based on that understanding, and application of key ideas. Clusters are groups of related Standards for each domain. Also included in the CCSSM are eight Standards for Mathematical Practice. These Standards describe the mathematical habits of mind that mathematically proficient students demonstrate in doing mathematics with understanding. The Practice Standards are

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

When students are actively involved in using the Practice Standards, they are learning meaningful, high-quality mathematics.

We suggest you work with your grade-level colleagues and use this book when you are studying the Standards, as you decide on the sequencing and clustering of the Standards, as well as the selection of appropriate instructional resources. Be sure to examine the content for the grade before and after the one you teach so you will understand what students should have learned and what they will be learning the next year. Keep in mind that implementation of the Standards and Practices is a process and may take time to do well. Your devotion to teaching the Standards will make a difference for students who will be learning to think, reason, and apply the mathematics you have taught them.

We hope you will find this book a helpful resource and a valuable companion as you work to help your students become successful mathematics learners.

Sincerely,

Linda M. Gojak

Ruth Harbin Miles

Letter to Elementary School Principals

Dear Elementary School Principal,

An instructional leader must clearly explain and help teachers understand that student success and achievement are the goals for implementing the Common Core Mathematics Standards and the Standards for Mathematical Practice. The role of the leader is not only to promote the Standards but also to ensure the Content Standards are taught and the Standards for Mathematical Practice are achieved in every classroom. As an instructional leader, a principal must help teachers engage in professional learning to study both the Content and the Practice Standards they will be teaching. Teachers will need guidance to understand the depth and the sequencing of each Standard as well as the content before and after their grade levels. Leaders must help teachers understand that the Common Core Mathematics Standards have the prospect of ensuring equity and access to high-quality mathematics for every student.

The Common Core Standards for Mathematics define what students should understand and be able to do in grades 3–5. Implemented properly, these Standards lay the foundation for the concepts and skills students will be expected to know in grades 6–12. Included in the Common Core Standards for Mathematics document are domains, clusters, and standards. The domains for 3–5 mathematics include five broad topics: Operations and Algebraic Thinking, Number and Operations in Base Ten, Number and Operations—Fractions, Measurement and Data, and Geometry. The Standards under each domain stress conceptual understanding, skills, and applications of key mathematical ideas. Clusters are groups of related Standards for each domain.

Also included in the Common Core Standards document are eight Standards for Mathematical Practice. These Standards describe the mathematical habits of mind that mathematically proficient students demonstrate in doing mathematics with understanding. The Practice Standards are

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

When students are actively involved in using the Practice Standards, they are learning meaningful, high-quality mathematics.

The Common Core Mathematics Companion: The Standards Decoded, Grades 3–5*: What They Say, What They Mean, How to Teach Them* is designed to support teachers in their learning and implementation of the Common Core Mathematics Standards. The book focuses on the critical ideas of 3–5 mathematics, including a meaningful explanation of each Standard along with effective teaching practices and learning activities. Mathematics vocabulary and suggested teaching materials are highlighted for each Standard. The book is not only a reference but a guide to helping teachers more deeply understand all aspects of the Standards.

Elementary schools, professional learning communities, individual classroom and special education teachers will all have different knowledge, various skills, and distinct ideas about using *The Common Core Mathematics Companion: The Standards Decoded, Grades* 3–5*: What They Say, What They Mean, How to Teach Them*. You may wish to supply every teacher with a personal copy of the book for use as a school-wide initiative or book study. Such a study will help improve both content knowledge and understanding of the mathematics teachers are expected to teach. Providing the opportunity for teachers to engage and use the book in grade-level planning with colleagues will allow teachers to dig deeply into the Standards. Use of this resource will add cohesiveness and

consistency, ensuring all grades 3–5 students will benefit from similar instruction. Be sure to invite teachers to bring this resource to all planning and professional development work. You may even want teachers to start or end a meeting with a lesson they've planned based upon the suggestions and strategies found in this reference guide. As a result of the book study, *The Common Core Mathematics Companion: The Standards Decoded, Grades 3–5: What They Say, What They Mean, How to Teach Them* will influence professional practice at both the classroom and school levels and will help transform instruction.

Sincerely,

Linda M. Gojak

Ruth Harbin Miles

Introduction

A Brief History of the Common Core

Contrary to popular belief, academic standards are not new. In fact they have been around for more than 25 years. The first set of curriculum specific standards, *The Curriculum and Evaluation Standards for School Mathematics*, was released by the National Council of Teachers of Mathematics in 1989, followed by an updated set of standards, *Principles and Standards for School Mathematics*, in 2000. Both of these documents provided a vision for K–12 mathematics by grade-level band. They also formed the foundation for most states' grade-level standards.

In April 2009 the National Governor's Association and the Council of Chief State School Officers met to discuss the creation of the Common Core State Standards Initiative. The purpose of such an initiative was to develop a set of common standards across states in order to balance the quality of mathematics instruction and learning. Following that meeting, the process of writing the Common Core Standards began. The Standards Development team, led by William McCallum, Phil Daro, and Jason Zimba, included mathematicians, mathematics educators, mathematics education researchers, and classroom teachers. The process included an open invitation for feedback, not only from mathematics educators and associations, including the National Council of Teachers of Mathematics, but also from the general public. This feedback was considered and much of it was incorporated into the final document released in June 2010. Following the release of the Standards, individual states went through their own processes for reviewing, adopting, and, if necessary, ratifying the adoption of the Common Core State Standards.

The Common Core State Standards for Mathematics

"The Common Core State Standards are a clear set of shared goals and expectations for the knowledge and skills students need in English language arts and mathematics at each grade level so they can be prepared to succeed in college, career, and life" (www.corestandards.org/about-the-standards/frequently-asked-questions/#faq-2303).

The Common Core State Standards for Mathematics (CCSSM) include two critical components of learning mathematics. The Content Standards explicitly outline the mathematics we want students to know and be able to do at each grade level. The Content Standards of the Common Core are fewer in number than most previous state standards. At the same time, the expectation is that students will develop deeper understanding of that content so less time is spent on reteaching from year to year. Additionally, the Standards were carefully constructed to show connections among ideas at a grade level as well as vertical progressions across grades. For example, you will find that the Standards in Grade 3 develop from the mathematical work that students have completed in previous grades. Similarly, the Standards in Grade 4 develop from work completed in grades K–3. Thus it is important for teachers to be knowledgeable of the Standards not only at the level they are teaching but also at the preceding grade level and the following grade level.

The second group of Standards, the Standards for Mathematical Practice, describes the habits of mind that students should develop as they do mathematics. These eight Standards are the same across all grade levels, K–12. As teachers plan mathematics lessons, they should consider how students will use the Practices in learning and doing mathematics.

The Common Core Standards *are not* a curriculum. Decisions about mathematics programs, textbooks and materials, sequencing topics and units, and instructional frameworks are left for local districts to make. They do not tell teachers how to teach. It is important to remember the Standards describe what students need to know and be able to do. Schools and teachers know best how to help students reach both the Content and the Practice Standards.

The Common Core Standards *do not* dictate specific assessments. Some states will be using assessments developed by PARCC (Partnership for Assessment of Readiness for College and Careers) or SBAC (Smarter Balanced Assessment Consortium). Others will develop and use their own assessments. Other facts and information can be found at http://www.corestandards.org.

Instructional Shifts

While the Standards do not call for a particular instructional model or philosophy, they are based on the best of existing standards. What is different is that they call for specific instructional shifts: *focus*, *coherence*, and *rigor*.

Focus: The Content Standards call for greater focus on fewer topics. An examination of the mathematics standards of high-performing countries indicate that fewer, more focused topics at a grade level allow students to deepen their understanding of the

mathematics and gain a stronger foundation for ongoing study of mathematics. Within the Standards, the major mathematical work of each grade level has been identified (www.corestandards.org). That means that not all of the content within a grade is emphasized equally among the Content Standards. The list of Content Standards for a grade is not linear, nor is it a checklist. Some clusters require greater emphasis than others. They take more time for students to master with depth of understanding. The major work of grades 3–5 includes multiplication and division of whole numbers and foundational understanding of fractions, leading to work in all operations with fractional numbers. This includes developing concepts, skills, and problem solving. This means the majority of instructional time in grades 3–5 (65% to 85%) should be spent on these mathematical topics. This does not mean that other Standards should be skipped. Rather, the supporting Standards should be taught to connect mathematical ideas among the essential Standards. The additional Standards provide students with experiences that will be foundational to work in future grades. Neglecting material will leave gaps in student skill and understanding (see the tables on pages 264–266).

Coherence: Many of us learned mathematics as a set of disconnected topics, with much of our skill based on tricks ("Ours is not to reason why, just invert and multiply!") or mnemonic devices ("Please Excuse My Dear Aunt Sally"). In reality, mathematics is a coherent body of knowledge made up of topics that are connected and build on each other. The call for coherence in the Content Standards ensures that there are carefully constructed progressions from grade to grade so students build new understandings on the foundations built in previous years. Each Standard is not a new topic, but an extension of previous learning. In addition to the progressions across grade levels, the Standards incorporate specific connections within a grade level. For example, as students develop conceptual understanding of multiplication and division, the relationship of these operations to each other is consistently reinforced through building conceptual understanding, procedural skills, and applying these understanding and skills to various contexts.

Rigor: The final instructional shift, rigor, refers to how we support students in developing deep understanding of each Standard. Understanding does not develop by assigning more worksheets or more difficult examples and problems. Rather, it calls for instructional practice that balances conceptual understanding, procedural skills, and applying mathematical ideas to a variety of contexts.

The following descriptions of each component of rigor come from www.corestandards.org.

Conceptual understanding: The Standards call for conceptual understanding of key concepts such as multiplication and division. Students must be able to access concepts from a number of perspectives in order to see mathematics as more than a set of rules or procedures.

Procedural skills and fluency: The Standards call for speed and accuracy in calculation. Students must practice core skills, such as basic facts and multiplication and division computation, in order to have access to more complex concepts and procedures. Fluency is built upon conceptual understanding and, with elementary children, through the development of ideas through representations using concrete materials, pictures, numbers, and words.

Application: The Standards call for students to use mathematics in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural fluency.

Major Work of Grades 3–5

To help drive the focus of the Standards, at least 65% and as much as 85% of instructional time should focus on the major work for each grade level. Areas of major work include:

Grade 3: Represent and solve problems involving multiplication and division; understand properties of multiplication and the relationship between multiplication and division; multiply and divide within 100; solve problems involving the four operations, and identify and explain patterns in arithmetic; develop understanding of fractions as numbers; solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects; geometric measurement: understand concepts of area and relate area to multiplication and to addition.

Grade 4: Use the four operations with whole numbers to solve problems; generalize place value understanding for multi-digit whole numbers; use place value understanding and properties of operations to perform multi-digit arithmetic; extend understanding of fraction equivalence and ordering; build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers; understand decimal notation for fractions, and compare decimal fractions.

Grade 5: Understand the place value system; perform operations with multi-digit whole numbers and with decimals to hundredths; use equivalent fractions as a strategy to add and subtract fractions; apply and extend previous understandings of multiplication and division to multiply and divide fractions; geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

Additional information on the focus for each grade level can be found in the Resources at the end of this book.

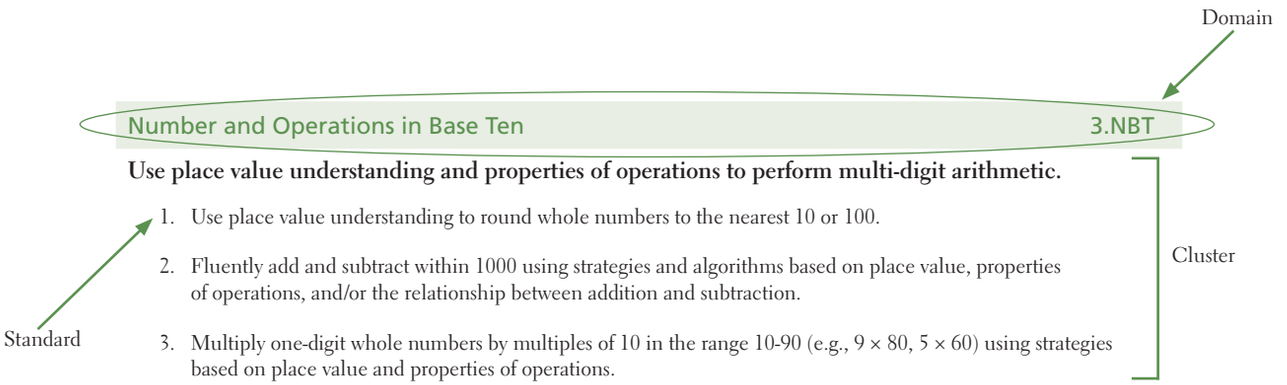
Common Core Word Wall

The language of the Common Core differs from traditional standards. Familiarity with section names and their functions will help you to make the best use of the Common Core Standards.

Standards define what students should understand and be able to do.

Clusters summarize groups of related Standards. Note that Standards from different clusters may sometimes be closely related, because mathematics is a connected subject.

Domains are larger groups of related Standards. Standards from different domains may sometimes be closely related.



Source: Common Core State Standards for Mathematics (www.corestandards.org)

As districts develop units of study for a grade level, careful consideration should be given to the order and connection among topics and Standards. For example, as third graders develop an understanding of the meaning of multiplication (3.OA.A.1), they use area models for multiplication to solve problems (3.OA.A.3), extend this understanding to multiply by multiples of 10 (3.NBT.A.3), and use area models to explore finding area of rectangular figures (3.MD.C.5,6,7).

The Common Core Standards for Mathematical Practice

The Common Core Standards for Mathematical Practice describe eight habits of mind teachers must incorporate into classroom instruction to help students develop depth of understanding of critical mathematical concepts. The mathematical Practices are not intended to be taught in isolation but should be integrated into daily lessons. Some lessons may focus on developing one or two of these Standards, and others may incorporate seven or all eight Standards. Note that you do not "teach" these Standards. Rather, they are the type of mathematical thinking and doing that we want students to practice as they are developing mathematical understanding.

Throughout the following chapters, we have included examples of mathematical practice that can be used in each cluster. These are not meant to limit lessons to using only those Practices. They are examples of key practices that should be included in lessons around that particular cluster. It is likely that you will use all of the Practices throughout the cluster and domain.

These eight Practices, briefly explained on the following page, are essential for student success. If students are actively engaged in using the eight Practices, they are learning rigorous, meaningful mathematics.

SFMP 1. Make sense of problems and persevere in solving them.

Students work to understand the information given in a problem and the question that is asked. They plan a solution path by choosing a strategy they can use to find a solution, and check to make sure their answer makes sense. As students in grades 3–5 work to make sense out of multiplication and division of whole numbers and fractions in these grades, using materials to solve problems helps them to develop conceptual understanding that leads to procedural fluency.

SFMP 2. Reason abstractly and quantitatively.

Students make sense of quantities and their relationships in problem situations. They develop operational sense by associating contexts to numbers (thinking about 4×7 in a context of 4 baskets with 7 eggs in each basket) and associate mathematical meaning to given contexts (having 4 baskets of 7 eggs can be expressed as 4×7). Modeling problem situations with concrete materials will help students to understand the meaning of multiplication and division and build a foundation for work with fractions.

SFMP 3. Construct viable arguments and critique the reasoning of others.

Students in grades 3–5 should have many opportunities to explain their thinking and justify and communicate their conclusions both orally and in writing. Listening to others and finding how their strategies are similar may take prompting questions from the teacher such as "Why do you think that works?" or "How is your method the same as . . . ?" Mathematical discussions should be a common expectation in mathematics lessons. It will take time for students to become comfortable explaining their thinking, but this will develop over time. Explaining one's thinking helps to develop deeper conceptual understanding.

SFMP 4. Model with mathematics.

Students use various representations, models, and symbols to connect conceptual understanding to skills and applications. As students work with the big ideas of grades 3–5, they should represent mathematics situations using objects, pictures, numbers, and words. Problem solving strategies such as draw a picture, make a list, find a pattern, and write an equation have explicit connections to representations and models and can be developed at the same time.

SFMP 5. Use appropriate tools strategically.

Students consider the available tools when solving a mathematics problem. Representations such as making equal groups, arrays, and area models will help students to see the connections between multiplication and division as well as the importance of place value in understanding these operations. Bar models, area models, and the number line will help students to understand fraction number concepts. A variety of concrete materials such as cubes, tiles, straws and rubber bands, fraction bars, and physical number lines will support students in these representations.

SFMP 6. Attend to precision.

Students communicate precisely with others. Students in grades 3–5 explain their thinking using appropriate mathematical vocabulary. Students expand their knowledge of mathematical symbols that should explicitly connect to vocabulary development.

SFMP 7. Look for and make use of structure.

Students look closely to find patterns and structure in their mathematics work. For example, students begin their work with fractions using unit fractions, which helps them to better understand the meaning of the numerator and the denominator. They extend their understanding of unit fractions to other common fractions as they develop a sense of equivalence and addition and subtraction of all fractions including mixed numbers. The relationship between multiplication and division of whole numbers extends to work with fractions.

SFMP 8. Look for and express regularity in repeated reasoning.

Students notice when calculations are repeated and begin to make generalizations. By recognizing what happens when multiplying or dividing tens or hundreds, students extend that understanding to more difficult problems. Although this Standard mentions shortcuts, it should be noted that shortcuts are only appropriate when students discover them through making generalizations and understand why they work.

Effective Teaching Practices

Quality mathematics teaching is a critical key for student success. In *Principles to Actions* (2014), the National Council of Teachers of Mathematics outlines eight valuable teaching practices every teacher should incorporate to guarantee student achievement. These eight research-informed practices briefly explained below provide a foundation for effective common core mathematics teaching and student learning.

1. Establish mathematics goals to focus learning.

Establishing learning goals sets the stage and helps to guide instructional decisions. Teachers must keep in mind what is to be learned, why the goal is important, where students need to go (the trajectory), as well as how learning can be extended. Students must clearly understand the purpose of each lesson beyond simply repeating the Standard.

2. Implement tasks that promote reasoning and problem solving.

Implementing tasks that promote reasoning and problem solving provides opportunities for students to engage in exploration and encourages students to use procedures in ways that are connected to conceptual understanding. The tasks teachers choose should be built on current student understandings and have various entry points with multiple ways for the problems to be solved.

3. Use and connect mathematical representations.

Using and connecting representations leads students to deeper understanding. Different representations, including concrete models, pictures, words, and numbers, should be introduced, discussed, and connected to support students in explaining their thinking and reasoning.

4. Facilitate meaningful mathematical discourse.

Facilitating meaningful student mathematical conversations provides students with opportunities to share ideas, clarify their understanding, and develop convincing arguments. Talking and sharing aloud can advance the mathematical thinking of the whole class.

5. Pose purposeful questions.

Posing purposeful questions reveals students' current understanding of a concept and encourages students to explain, elaborate, and clarify thinking. Asking good questions makes the learning of mathematics more visible and accessible for student examination.

6. Build procedural fluency from conceptual understanding.

Building procedural fluency from conceptual understanding based on experiences with concrete representations allows students to flexibly choose from a variety of methods to solve problems.

7. Support productive struggle in learning mathematics.

Supporting productive struggle in learning mathematics is significant and essential to learning mathematics with understanding. Productive struggle allows students to grapple with ideas and relationships. Giving young students ample time to work with and make sense out of new ideas is critical to their learning with understanding.

8. Elicit and use evidence of student thinking.

Eliciting and using evidence of student thinking helps teachers access learning progress and can be used to make instructional decisions during the lessons as well as help to prepare what will occur in the next lesson. Formative assessment through student written and oral ideas are excellent artifacts to assess student thinking and understanding.

How to Use This Book

The purpose of this book is to help teachers more deeply understand the mathematical meaning of each cluster and Standard within the five domains of grades 3–5. We want this book to be your toolkit for teaching the mathematics Standards, and we have left ample space for you to take notes and add ideas and other resources you have found to be helpful.

You will find each part of this book includes one domain and begins with an overview of how the domain progresses across third, fourth, and fifth grades. A list of helpful materials, reproducibles, and key vocabulary from the domain is included in the overview as well.

We track each domain across third, fourth, and fifth grades with a page for each cluster and the Standards within that cluster. A description of the cluster and how the Standards for Mathematical Practice can be incorporated into your teaching of the cluster concepts follows. Because the Standards are intentionally designed to connect within and across domains and grade levels, a list of related Standards is included in the cluster overview. We suggest that as you prepare work on a cluster you look at these Standards to have a better idea of the mathematics students learned in previous grades and where they are going in future grades. A list of all of the Standards is found in the Quick Reference Guide at the beginning of the book.

Each Standard within a cluster is explained with an example of *What the TEACHER does* to work with that Standard in the classroom followed by a description of *What the STUDENTS do*. It is important to note that most Standards will take several days, and you should be connecting conceptual understanding across Standards and domains as you teach for understanding.

Addressing student misconceptions and common errors in developing student understanding of a concept concludes the contents for each Standard.

Each cluster ends with a template for planning instruction for that cluster. At the end of each domain you will find a sample planning page based on one Standard for that domain. Also included are planning page templates for each cluster within the domain for you to duplicate and use in your planning.

In the Resources section, you will find tables that are fundamental to the Operations and Algebraic Thinking and Number and Operations in Base Ten domains. You will also find reproducibles for key materials. These are designed to be samples, and we encourage you to use them or redesign them to best meet the needs of your students. A list of our favorite resource books and high-quality online resources that are particularly useful to developing mathematical ideas in grades 3–5 are also included.

We believe that this can become your common core bible! Read it and mark it with questions, comments, and ideas. We hope that it will help you to use these Standards and good teaching practice to lay the essential foundation that will ensure your students success not only in your grade, but in all of their future study of mathematics.

Reflection Questions

1. How are the three instructional shifts called for by the Common Core similar to your current instructional practice? What is conceptual understanding? How is it different from procedural skills? What do you need to consider to teach for conceptual understanding? How can you connect conceptual understanding to help students develop procedural skills? How does the information in Table 1, page 254, on problem situations support the development of conceptual understanding?

2. The Standards for Mathematical Practice describe the habits of mind that students need for thinking about and doing mathematics. While not every Standard will be in every lesson, select one Standard at your grade level and consider some ways you can incorporate these Practices in a lesson for that Standard. How will these Practices provide you with information about student understanding? How will this help you to better assess students? How will this information help you in planning lessons?

3. The Effective Teaching Practices describe specific actions that teachers must consider in planning and implementing lessons and assessing student performance. How are these Practices connected? Work with colleagues to plan a lesson that employs all of these Practices. What needs to be considered as you consider goals for the lesson? How can you modify a traditional task so that it promotes reasoning and problem solving? What representations will help students more deeply understand the concept? What questions will you ask students? How will you connect the conceptual understanding to build procedural fluency? What questions will support students who are working to make sense of a new idea? What kind of information will you look for to help inform you instruction? (For more information on the Effective Teaching Practices, go to www.nctm.org.)

Part 1

Operations and Algebraic Thinking

Operations and Algebraic Thinking

Domain Overview

GRADE 3

The major work of this domain in Grade 3 is to develop students' conceptual understanding of multiplication and division by using concrete materials to model multiplication and then relate their understanding of multiplication to division. Multiplication problem situations provide a context for understanding multiplication as finding the total number of items given a number of equal groups and the number of items in each group. Division problem situations develop the meaning of division and how it is related to multiplication. When you know the total number of items and the number of groups, you can determine how many items in a group, or, when you know the total number of items and the number of items in a group, you can find the number of groups. All of these activities culminate in the expectation that students will demonstrate fluency with multiplication and division within 100 using single-digit factors.

GRADE 4

Students in Grade 4 continue to solve problems using the four operations with whole numbers. New to this grade level are problem situations that involve multiplicative comparisons. Students become familiar with factors and multiples and how they relate to prime and composite numbers. They work in a variety of contexts to generate and analyze patterns.

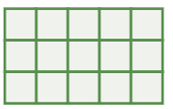
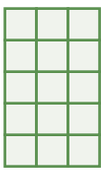
GRADE 5

In preparation for the Expressions and Equations domain in grades 6–8, fifth graders begin to explore, interpret, and evaluate numerical expressions. Work with patterns that began in Grade 4 extends to generating patterns, forming ordered pairs, graphing on a coordinate plane, and then analyzing the graphical representations.

SUGGESTED MATERIALS FOR THIS DOMAIN

3	4	5	
✓	✓	✓	Hundreds chart (Reproducible 1)
✓	✓	✓	Chips, counters
✓	✓	✓	Cups, containers, other objects to represent "groups"
✓	✓	✓	Place value chart to hundreds (Reproducible 2)
✓	✓	✓	Square tiles
✓	✓	✓	Grid paper (Reproducible 3)
✓	✓	✓	Pattern blocks
✓	✓	✓	Number cards (such as a deck of playing cards)

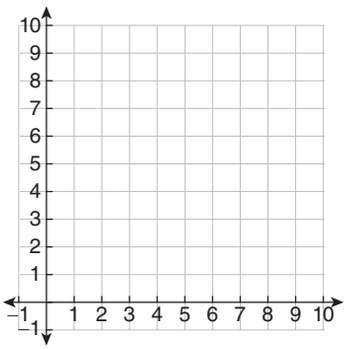
KEY VOCABULARY

3	4	5	
✓	✓	✓	**add** to combine or join together related words: *add, and, plus, join, put together, (+)*
✓	✓	✓	**addend** any of the numbers added to find a sum
✓	✓	✓	**area model** a concrete model for multiplication or division made up of a rectangle. The length and width represent the factors and the area represents the product.  3 × 5  5 × 3
✓	✓	✓	**array model** a concrete model for multiplication in which items are arranged in rows and columns. Each row (or column) represents the number of groups and each column (or row) represents the number of items in a group.  5 × 4 = 20 5 rows of 4 = 20
✓	✓	✓	*** associative property of multiplication** an extension of the commutative property; to change the order and group two factors to find convenient products (such as 10) in order to make the multiplication easier. Students may begin to use parentheses at this level. 7 × 8 × 5 = 7 × (8 × 5) = 7 × 40 = 280
✓	✓	✓	*** commutative property of multiplication** reversing the order of the factors does not change the product 8 × 5 = 40 and 5 × 8 = 40 therefore the product of 8 × 5 = 5 × 8

(Continued)

(Continued)

KEY VOCABULARY

3	4	5	
	✓	✓	**comparison model** a multiplication or division situation in which one number is a multiple of the other *Example:* Maya has 5 marbles. Alexa has 3 times as many. How many marbles does Alexa have?
	✓	✓	**compose** put a number together using other numbers 1 + 9, 2 + 8, 3 + 7, 4 + 6, 5 + 5, 1 + 2 + 3 + 4 are ways to compose 10
	✓	✓	**composite number** a number that has more than two factors
		✓	**coordinate plane** a plane determined by a horizontal number line, called the *x*-axis, and a vertical number line, called the *y*-axis, intersecting at a point called the origin. Each point in the coordinate plane can be specified by an ordered pair of numbers. 
✓	✓	✓	**decompose** separate a number into parts using other numbers 8 can be decomposed into 4 + 4, 3 + 5, 2 + 2 + 2 + 2
✓	✓	✓	*** distributive property** multiplying a sum by a given number is the same as multiplying each addend by the number and then adding the products 6 × 9 = 54 6 × (5 + 4) = (6 × 5) + (6 × 4) = 30 + 24 = 54 The distributive property says that if a, b, and c are real numbers, then: $a \times (b + c) = (a \times b) + (a \times c)$
✓	✓	✓	**division** sharing a number into equal groups and finding the number of groups or the number of items in each group
✓	✓	✓	**equal groups model (measurement division)** a division model in which the total number of items and the number of items in each group is known and the number of groups that can be made is the unknown. *Example:* I have 48 peanuts. I want to put 8 peanuts in a cup. How many cups will I need?
✓	✓	✓	**equation** a mathematical sentence in which one part is the same or equal to the other part 3 + 5 = 8 12 − 7 = 5 11 = 8 + 3 6 = 9 − 3
✓	✓	✓	**estimate** to make an approximation or calculate using closer or easier numbers
	✓	✓	**evaluate** find the numerical value of mathematical expression
✓	✓	✓	**expression** one or more mathematical symbols that represent a number or quantity examples of expressions 3 × 6 4 + 7 × 3 8
✓	✓	✓	**fact family** a set of related mathematics facts, such as 3 × 5 = 15 15 = 5 × 3 15 ÷ 3 = 5 3 = 15 ÷ 5
✓	✓	✓	**factor** one of the numbers multiplied to find a product

KEY VOCABULARY

3	4	5	
	✓	✓	**factor pair** a pair of numbers that when multiplied give a product; for example, 1 and 15, 3 and 5 are factor pairs for 15
✓	✓	✓	**fair share model (partitive division)** a division model in which the total number and the number of groups is known and the number of items in each group is unknown *Example:* I have 48 peanuts and want to put them into 6 cups. If I put the same number of peanuts into each cup, how many peanuts will be in each cup?
✓	✓	✓	*** identity property of multiplication** any number multiplied by 1 equals the number $3 \times 1 = 3$ $1 \times 3 = 3$
✓	✓	✓	**measurement division (equal groups model)** a division model in which the total number of items and the number of items in each group is known and the number of groups that can be made is the unknown *Example:* I have 48 peanuts. I want to put 8 peanuts in a cup. How many cups will I need?
✓	✓	✓	**missing factor** the unknown factor when a product and one factor are known $4 \times \square = 32$ The missing factor is 8.
	✓	✓	**multiple** the result of multiplying a whole number by other whole numbers multiples of 5 are 0, 5, 10, 15, 20, 25, 30
✓	✓	✓	**multiplication** a mathematical operation in which a number is added to itself a specific number of times; one factor tells the number of groups or sets, the other factor tells the number of items in a group or set and the result, or product, tells the total number of items $3 \times 5 = 15$ 3 groups with 5 in each group would give a total of 15
✓	✓	✓	**number line** a line used to show the position of a number in relation to zero and other numbers
		✓	**ordered pair** a pair of numbers that gives a location on a coordinate plane. The first number is the *x* coordinate and the second number is the *y* coordinate.
✓	✓	✓	**partitive division (fair share model)** a division model in which the total number and the number of groups is known and the number of items in each group is unknown *Example:* I have 48 peanuts and want to put them into 6 cups. If I put the same number of peanuts into each cup, how many peanuts will be in each cup?
✓	✓	✓	**pattern** set of numbers or objects that can be described by a specific rule
	✓	✓	**prime number** a number that has exactly two factors
✓	✓	✓	**product** the result when two numbers are multiplied
✓	✓	✓	**quotient** the result when two numbers are divided; the missing factor
✓	✓	✓	**remainder** amount left when two numbers are divided
✓	✓	✓	**round** to change a number to a less exact number that is more convenient for computation
✓	✓	✓	**strategy** a plan to find an answer or solve a problem that makes sense
✓	✓	✓	**sum** the result when two numbers are added
✓	✓	✓	**unknown** the quantity you are finding in a mathematics problem
✓	✓	✓	*** zero property of multiplication** any number multiplied by 0 equals 0 $8 \times 0 = 0$ (8 groups of 0 is 0) $0 \times 8 = 0$ (0 groups of 8 is 0)

*Students are not responsible for these vocabulary words; however, they should understand the mathematical concept.

Operations and Algebraic Thinking
3.OA.A.*

Cluster A

Represent and solve problems involving multiplication and division.

STANDARD 1

3.OA.A.1: Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5×7.

STANDARD 2

3.OA.A.2: Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as* $56 \div 8$.

STANDARD 3

3.OA.A.3: Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

[1] See Table 2 in the Resources, page 256.

STANDARD 4

3.OA.A.4: Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations* $8 \times ? = 48$, $5 = \_ \div 3$, $6 \times 6 = ?$

*Major cluster

Operations and Algebraic Thinking 3.OA.A

Cluster A: Represent and solve problems involving multiplication and division.
Grade 3 Overview

Third grade students explore the meaning of multiplication as finding the total number of objects (product) when they know the number of groups (factor) and the number of items in each group (factor). The relationship between multiplication and division helps students understand that when dividing, they are finding the number of groups (missing factor) when they know the total count (product) and the number of items in a group (factor), or finding the number of items in a group (missing factor) when they know the number of groups (factor) and the total count (product). Problem solving situations and activities that include a variety of representations showing equal-sized groups, arrays, and area models lay the foundation for multiplication and division of whole numbers.

Note that these Standards are not linear. It is important for students to understand the meaning of multiplication and division (3.OA.A1, 3.OA.A.2) through the use of problem situations (3.OA.A.3). As students demonstrate understanding they begin to relate models to symbolic notation (3.OA.A.4). The use of symbols for easier facts and relating the symbols to fact families should be happening as students continue to use models to solve problems with the more difficult facts.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students solve a variety of problems as contexts for learning what it means to multiply or divide. They use quantitative reasoning to determine what is happening when they multiply (given the number of groups and the number of items in a group, they find the total number of items) and divide (given the total number of items and the number of groups, they find the number of items in a group *or* given the total number of items and the number of items in a group, they find the number of groups). Constructing mathematical arguments to justify their reasoning and comparing their strategies with those of classmates helps students to make connections among ideas and between concrete models and numerical notations (expressions and equations). They use a variety of tools to model multiplication and division including sets, arrays, area models, and the number line to represent what is happening when they multiply or divide. Developing the mathematical vocabulary of multiplication and division (factor × factor = product and product ÷ factor = missing factor) helps students to explain their thinking not only about the individual operations but also how they are related to each other. The commutative, associative, and distributive properties lay the foundation for fluency with basic facts through looking at the structure of multiplication and division and provide students with strategies for solving problems. Students use patterns and repeated reasoning (multiplication by 0, 1, 5, and 10) to help them identify patterns and become fluent with basic facts.

Related Content Standards

2.OA.C.3 2.OA.C.4 4.OA.A.1 4.OA.A.2

Notes

STANDARD 1 (3.OA.A.1)

Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7.

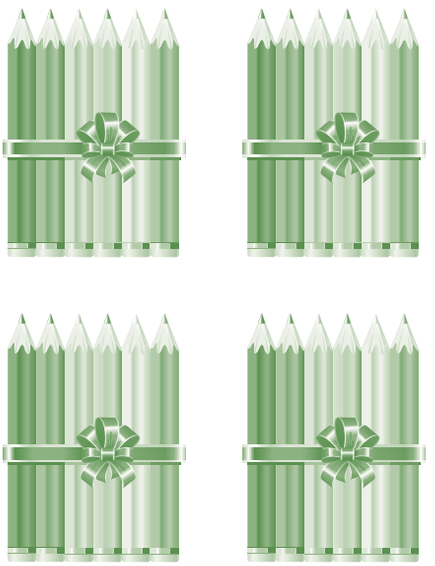
Students develop an initial understanding of multiplication of whole numbers by modeling situations in which there are a specific number of groups with the same number of items in each group. Unlike addition, in which each addend represents a certain number of items, in multiplication one factor represents the number of groups and the other factor represents the number of items in each group. The product represents the total number of items in all of the groups. Problem situations provide students contexts for using concrete materials.

New vocabulary includes *factor*, *equal groups*, *product*. The symbol × means groups of (or times) and 3 × 5 can initially be read as "3 groups of 5."

Example:

Mrs. Flack has 4 packages of pencils for the class. Each package contains 6 pencils. How many pencils does Mrs. Flack have?

This can be expressed as 4 × 6 or 4 groups of 6.



What the TEACHER does:

- Provide students with a variety of multiplication situations to model using concrete materials such as chips, counters, straws to represent the items, and cups, egg carton, paper to represent groups.
- Ask students to identify the number of groups and the number of items in each group and then the total number of items.

  There are 3 seats in the van and each seat can hold 4 people. How many people can ride in the van?

3 groups of 4 people

3 × 4 = 12

- Introduce multiplication terminology and symbols as students are ready. Add these terms and symbols to the class multiplication and division word wall.
  - factors, product, groups, times
  - ×
- Introduce students to numerical representations by writing equations that represent their work.

  3 × 4 = 12
- As students show understanding and can identify the number of groups, the number of items in a group, and connect that to the symbolic representation, progress to situations with pictures, numbers, and words.

What the STUDENTS do:

- Use concrete materials to model various multiplication situations.
- Identify the number of groups and the number of items in each group.
- Explain how they determined the total number of items.
- Connect representations to numeric expressions.
- Use pictorial representations for multiplication situations.
- Use appropriate vocabulary (*factor*, *product*, *times*, *groups of*) to describe their work.
- Write expressions and equations for their models and drawings.

Addressing Student Misconceptions and Common Errors

In previous work with addition, both addends represented the count or number of items that are joined for a total count. For example, 6 markers and 3 more markers give a total of 9 markers. In multiplication, one factor represents the number of groups, sets, or collections, and the other factor represents the number of items in each group, set, or collection. Students need multiple experiences identifying which factor represents the number of groups and which factor represents the number of items in each group. Early experiences with concrete models and pictures and explicit connections to the symbolic notation will not only help students to identify multiplication situations but will also support student understanding of division.

Notes

STANDARD 2 (3.OA.A.2)

Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.

Once students understand the meaning of multiplication in terms of finding the total number of items given the number of groups and the number of items in a group, division can be understood by thinking in terms of finding a missing factor (either the number of groups or the number of items in a group).

There are two distinct meanings of division. The first is the partitive (or fair share) meaning.

Example 1:

John has 32 crayons and 4 bags. If he wants to put the same number of crayons in each bag, how many crayons will he put in each bag?

In this case, John knows the total number of items (product) and the number of bags or groups (factor) and he is looking for the number of items to put in each bag (missing factor). This can be written as 4 × ____ = 32 or as the division expression 32 ÷ 4 = ____.

The other meaning of division is the measurement (or repeated subtraction) meaning.

Example 2:

John has 32 crayons. He wants to give 8 crayons to each person in his group. How many people are in John's group?

In this case, John knows the total number of items (product) and the number of items for each person (factor). He wants to find the number of people or groups of 8 he can make (missing factor). This can be written as ____ × 8 = 32 or as the division expression 32 ÷ 8 = ____.

It is not necessary for students to know the names of each division meaning; however, they should be able to identify the unknown in a given situation.

Type of Division	Number of Groups	Number of Items in a Group	Total Number of Items
Partitive	known	unknown	known
Measurement	unknown	known	known

What the TEACHER does:

- Provide a variety of division situations for students to model using concrete materials.
- Ask questions that support students in identifying information in the problem and connecting it to the division models. What do you know? (For example, the number of groups) What do you want to find out? (For example, how many in a group?)
- Present opportunities for students to explain their reasoning.
- Introduce division symbols and terminology. Add these terms to the class multiplication and division word wall.
  - ÷ , $\overline{)}$
  - factor, missing factor, product, divisor, dividend, quotient
- Introduce students to numerical representations by writing equations that represent their work.
  - 15 ÷ 5 = 3
  - $3\overline{)15}$
- As students show understanding and can identify the given information (number of groups or number of items in a in a group) and connect it to the symbolic representation, progress to situations with pictures, numbers, and words.

What the STUDENTS do:

- Use concrete materials to model various division situations.
- Identify the information given in the problem as well as the missing information.
  - **There are 18 marbles and each player needs 6 marbles. How many people can play?**
    - I know each player (group) has 6 marbles and there are 18 marbles. I want to find how many people (groups) can play. 18 ÷ 6 = 3
  - **There are 20 candies in a package. I have 4 candy cups. If I need to put the same number of candies in each cup, how many cups can I fill?**
    - I know there are 20 candies and I have 4 cups (groups). I want to find out how many candies I can put in each cup. 20 ÷ 4 = 5
- Explain how they determined the missing factor.
- Use pictorial representations for division situations.
- Connect representations to numeric expressions.

Addressing Student Misconceptions and Common Errors

Because multiplication is commutative (3 × 7 = 7 × 3), some students think that 21 ÷ 3 and 3 ÷ 21 mean the same thing. This is especially true the equations are written two different ways.

21 ÷ 3 and $3\overline{)21}$

Connecting concrete and pictorial models to both forms of division equations is essential to eliminating this misconception.

Students read $3\overline{)21}$ as 3 "goes into" 21. Although these words are commonly used, they do not reinforce the meaning of division. Getting students to read this as "3 divides 21" or "21 divided by 3" or "How many groups of 3 are in 21?" is a habit that should be developed early in division instruction.

The sharing model (How many in a group?) is often easier for students to recognize as division. The measurement model is more difficult. Students need to work with many problem situations for each type of division using concrete materials and drawing pictures.

STANDARD 3 (3.OA.A.3)

Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

[1] See Table 2 in the Resources, page 256.

Table 2 (page 256) provides problem situations for multiplication and division. These contexts provide important links to developing conceptual understanding of the meaning of multiplication and division. Begin with modeling equal group situations and progress to array and area situations. Note that comparison situations do not need to be introduced until Grade 4.

It is important for students to have many opportunities to use concrete materials to model the situations and identify the number of groups and the number of items in a group. Only after evidence of students' understanding using concrete models should they begin to draw pictures and use the number line as representations of multiplication.

Once students demonstrate understanding with multiplication situations, use connected division examples in which students identify the total number of objects and explain whether they know the number of groups or the number of items. Later, provide array and area situations with multiplication examples in which students find the total number of items, and make connections to division examples in which students need to find the missing factor (number of columns or number of rows).

What the TEACHER does:

- Provide students with a variety of equal group multiplication situations and have them model each situation to solve the problem.
  - Students identify the factor that represents the groups and the factor that represents the total number in a group.
  - Give students opportunities to discuss their representations and explain their thinking.

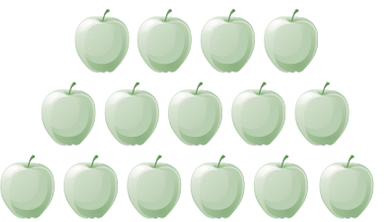
Example:

There are 5 bags of apples. Each bag contains 3 apples. How many apples are there?

- Provide students with related partitive (fair share) division situations in which they identify the given information (total number of items, number of groups) and find the number of items in each group.

Example:

I have 15 apples and 5 bags. I want to put the same number of apples in each bag. How many apples are in each bag?



- Provide students with related measurement division situations in which they identify the given information (total number of items and number of items in each group) and find the number of groups.

(continued)

What the TEACHER does (continued):

Example:

I have 15 apples and I want to put 3 apples in each bag. How many bags do I need?

- After a variety of experiences representing equal group problems with concrete materials and pictorial representations, make connections to the written symbols for multiplication and division, including missing factor notation.
  - 5 groups of 3 is 15 $5 \times 3 = 15$
  - $5 \times \square = 15$ $15 \div 5 = \square$ $5\overline{)15}$ $\quad 3$
  - $\square \times 3 = 15$ $15 \div 3 = \square$ $3\overline{)15}$ $\quad 5$
  - Continue with similar situations using array representations and following the procedure as described for equal group situations (see Table 2, page 256).

What the STUDENTS do:

- Work collaboratively using concrete materials to represent multiplication problems.
- Identify the information given in the problem and explain their thinking using multiplication vocabulary (*groups*, *total*, *factor*, and *product*).
- Use mathematical symbols to represent the problem situation.
- Work collaboratively to model related division problems.
- Identify the information given in the problem and explain their thinking using division vocabulary (*factor—number of groups*, *number of items in a group*, and *product—total number of items*).
- Use mathematical symbols to represent the problem situation using either multiplication notation with a missing factor or division notation.
- Explain their reasoning for each problem situation.

Addressing Student Misconceptions and Common Errors

Students who have trouble identifying information in a problem situation (which number represents the total, the number of groups and/or the number of items in a group) need more experience making explicit connections between their representations (concrete models or pictures) and determining the number of groups **or** the number of items in a group.

STANDARD 4 (3.OA.A.4)

Determine the unknown whole number in a multiplication or division equation relating three whole numbers.
For example, determine the unknown number that makes the equation true in each of the equations
$8 \times ? = 48$, $5 = \_\_ \div 3$, $6 \times 6 = ?$

Up to this point, students have had many experiences solving multiplication and related division problems using concrete materials. As students develop conceptual understanding of these operations, including modeling and explaining situations, they now relate this understanding to symbolic notation, writing equations. Students may be writing equations for simpler facts while they continue to work with models and other representations for the more difficult facts. Focusing on the relationship between multiplication and division will help students develop fluency with related fact families.

What the TEACHER does:

- As students become familiar with equation notation connected to their concrete or pictorial representations, give them isolated equations to talk about and describe what the numbers in each equation represent. For example, $3 \times 4 = 12$ represents 3 groups of 4 items gives a total of 12 items.
- Include opportunities for students to explore writing equations with missing factors.

  $\square \times 4 = 12$ means some number of groups of 4 items equals 12 items, and connecting the multiplication missing factor equations to related division equations $12 \div 4 = \square$.
- Expect students to build fact families and explain the relationship among the facts, first using multiplication facts with missing factors and then using division facts.

What the STUDENTS do:

- Write and read related multiplication equations and equations with missing factors.
  - $3 \times 5 = 15$ $\quad 15 = 3 \times 5$
  - $3 \times \square = 15$ $\quad 15 = 3 \times \square$
  - $\Delta \times 5 = 15$ $\quad 15 = \Delta \times 5$
- Relate missing factor multiplication equations to division equations using both the ÷ and $\overline{)}$ symbols for division.
  - $3 \times 5 = 15$ $\quad 15 = 3 \times 5$ $\quad 15 \div 3 = 5$ $\quad 3\overline{)15}$ $\quad 15 \div 5 = 3$ $\quad 5\overline{)15}$
  - $3 \times \square = 15$ $\quad 15 = 3 \times \square$ $\quad 15 \div 3 = \square$ $\quad 3\overline{)15}$
  - $\Delta \times 5 = 15$ $\quad 15 = \Delta \times 5$ $\quad 15 \div 5 = \Delta$ $\quad 5\overline{)15}$

Addressing Student Misconceptions and Common Errors

Now that students are working more frequently with numeric equations for multiplication and division, reinforce accurate reading of the equations. $15 \div 3$ and $3\overline{)15}$ should both be read as "15 divided by 3" or "3 divides 15."

In algebra, the use of a symbol in one problem cannot represent a different number in a different but related situation. When writing missing factor equations, be sure to use different symbols for the missing factor that represents the number of groups and the missing factor that represents the total number in a group or the total number of items.

For example: $15 \div 3 = x$ $\quad 15 \div 5 = y$

Notes

Operations and Algebraic Thinking
3.OA.B*

Cluster B

Understand properties of multiplication and the relationship between multiplication and division.

STANDARD 5

3.OA.B.5: Apply properties of operations as strategies to multiply and divide.[2] *Examples: If* $6 \times 4 = 24$ *is known, then* $4 \times 6 = 24$ *is also known. (Commutative property of multiplication.)* $3 \times 5 \times 2$ *can be found by* $3 \times 5 = 15$*, then* $15 \times 2 = 30$*, or by* $5 \times 2 = 10$*, then* $3 \times 10 = 30$*. (Associative property of multiplication.) Knowing that* $8 \times 5 = 40$ *and* $8 \times 2 = 16$*, one can find* 8×7 *as* $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$*. (Distributive property.)*

[2] Students need not use formal terms for these properties.

STANDARD 6

3.OA.B.6: Understand division as an unknown-factor problem. *For example, find* $32 \div 8$ *by finding the number that makes* 32 *when multiplied by* 8.

*Major cluster

Operations and Algebraic Thinking 3.OA.B

Cluster B: Understand properties of multiplication and the relationship between multiplication and division.
Grade 3 Overview

As students have a variety of experiences solving problems and modeling multiplication and division situations with one-digit factors, they explore the properties of multiplication, develop strategies based on these properties, and use the properties to build their understanding of the relationship between multiplication and division. Properties include the commutative and associative properties, the identity element for multiplication, and the zero property. These properties can be connected to earlier work with addition. The distributive property will help students to develop efficient strategies for multiplication—not only for basic facts but also for more complex multiplication examples. It is also a foundational property for future work with algebra.

Standards for Mathematical Practice
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics and SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students use concrete models, pictures, words, and numbers to justify their ideas showing the patterns they find in multiplication and division. Developing the precise language of multiplication and division helps students to explain their thinking not only about the individual operations but how they are related to each other. The commutative, associative, and distributive properties lay the foundation for fluency with basic facts through looking at the structure of multiplication and division and providing students with strategies for solving problems. Students look for and describe patterns they notice as they work with multiplication and division facts.

Related Content Standards

4.NBT.B.5 4.NBT.B.6 5.NBT.B.6 5 NBT.B.8

STANDARD 5 (3.OA.B.5)

Apply properties of operations as strategies to multiply and divide.[2] Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)

[2] Students need not use formal terms for these properties.

As third graders explore and develop conceptual understanding of multiplication and division, they recognize the structure of multiplication by noticing patterns and making generalizations about multiplication and division applying a variety of properties. These properties are not taught in isolation, but rather should be developed and discussed as a part of carefully related student experiences. Note that instruction does not stop to "teach" the properties. Incorporate opportunities for students to use the properties to develop strategies and patterns to simplify what is happening when they multiply two numbers.

Providing students with multiple experiences to multiply with a factor of 1 will lead to a discussion of 1 as the identity element for multiplication. Multiplying a number by 1 does not change the number.

$\square \times 1 = \square$ $\quad 1 \times \square = \square$ $\quad 1 \times 7 = 7$ $\quad 7 \times 1 = 7$

$\square = \square \times 1$ $\quad \square = 1 \times \square$ $\quad 7 = 1 \times 7$ $\quad 7 = 7 \times 1$

The zero property of multiplication states that if one of the factors is zero (I have zero groups or zero items in a group) the product is zero.

$\square \times 0 = 0$ $\quad 0 \times \square = 0$ $\quad 3 \times 0 = 0$ $\quad 0 \times 3 = 0$

$0 = \square \times 0$ $\quad 0 = 0 \times \square$ $\quad 0 = 3 \times 0$ $\quad 0 = 0 \times 3$

Although students worked with the commutative property of addition in earlier grades, the commutative structure of multiplication is different because factors represent two different quantities—one being the number of groups and the other being the number of items in each group. So, although the product for 6×3 and 3×6 is the same, the actual multiplication situations are not the same. One represents 6 groups of 3 and the other represents 3 groups of 6.

$a \times b = c$ and $b \times a = c$

$3 \times 5 = 15$ and $5 \times 3 = 15$

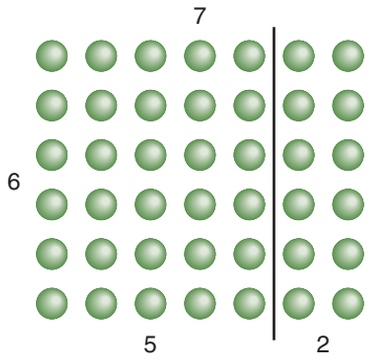
The associative property of multiplication shows that when multiplying three or more numbers, the product is always the same regardless of their grouping. That is, $(a \times b) \times c = a \times (b \times c)$. This property is particularly helpful in developing strategies for mental computation and decomposing factors to help students learn more difficult multiplication facts.

$(a \times b) \times c = a \times (b \times c)$ $\quad (7 \times 2) \times 5 = 70$ $\quad 7 \times (2 \times 5) = 70$

The distributive property should be explored in the context of composing and decomposing factors. Although the use of this property becomes much more formal in algebraic contexts, third graders who understand the distributive property can use it to help learn more difficult basic facts. Using concrete representations will help students to conceptually understand the distributive property rather than to learn it as a rule or formal procedure.

Example:

When multiplying 6 × 7, I can think of this as 6 groups of 7—or I can also decompose the 7 to think about it as 5 + 2 so I can show 6 groups of 5, which is 30, and 6 groups of 2, which is 12. Adding 30 and 12 is 42, so 6 × 7 = 42.



$6 \times 7 = 42$

$6 \times 5 = 30$ $6 \times 2 = 12$

$6 \times (5 + 2) = (6 \times 5) + (6 \times 2)$

What the TEACHER does:

- Present students with a variety of multiplication examples using concrete representations including equal group and array models. Ask students questions that lead them to conclude that the order of the factors does not affect the product. For example:

  Hank has 4 bags with 3 marbles in each bag.

  Denise has 3 bags with 4 marbles in each bag.

  - What is similar in these two situations?
  - What is different in these two situations?
  - What do you notice about the factors in these two problems?
  - What does each factor represent for Hank?
  - What does each factor represent for Denise?
  - What do you notice about the total number of marbles each person has?
  - Try this with two different factors. What do you notice about the product of these factors?
  - Do you think this will always be true?

- Provide students with opportunities to explore the identity element for multiplication. (*Note:* They do not need to know that 1 is the identity element; rather, they recognize that when I multiply a number times 1, the product is the number.) Have students investigate a variety of situations for multiplication by one and generalize the identity element.

  Karen has 1 package with 6 cupcakes. How many cupcakes in all?

  Marianne has 6 packages with 1 cupcake. How many cupcakes in all?

- Continue similar experiences with multiplication by zero.

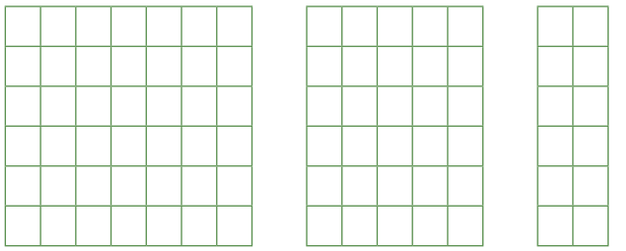
  Cathy is paid $4 an hour for mowing the lawn. It rains so she cannot mow today. How much will she get paid?
  $0 \times 4 = 0$

  Cathy volunteers to mow the lawn for her grandmother. It takes her 4 hours to mow the lawn. She does not ask for any money. How much will she get paid?
  $4 \times 0 = 0$

  (Although these situations may seem trivial, they do provide an example of context that demonstrates that when multiplying with 0 the product will be 0.)

- Provide examples for students to explore that model the associative property to help them discover strategies for making multiplication of several factors easier.
  - Find the product of $3 \times 8 \times 5$.
  - Find the product of $3 \times (8 \times 5)$, or multiply 8×5 and then multiply that product by 3.
  - What do you notice about these two problems?
  - What do you notice about the products?
  - Which is easier to do? Why?

- Introduce the distributive property by having students build a multiplication array for 6×7 using square tiles. Write the equation for this array. Ask them to divide the array into 2 pieces along a horizontal or vertical line, as shown below.



$6 \times 7 = 42$ $6 \times 5 = 30$ $6 \times 2 = 12$

$30 + 12 = 42$

(continued)

What the TEACHER does (continued):

- Collect the models and equations from the various ways students divided the array. Record the equations on the board. Facilitate a discussion that helps students to understand the relationship between the two figures. Ask
  - What are the factors in the first array?
  - How many tiles are in the first array?
  - How are the factors in the new arrays similar to the factors in the first array?
  - What do you notice about the total number of tiles in the new arrays?
- Provide students with a variety of opportunities similar to that above. Give students many opportunities to explore the distributive property and then connect it to the symbolic notation.

  $6 \times 7 = 42$ $6 \times (5 + 2) = 42$ $6 \times 5 + 6 \times 2 = 42$
- Encourage students to use models of the distributive property to help them find the product of facts with which they struggle.

What the STUDENTS do:

- Use concrete materials to model specific multiplication situations.
- Discuss patterns they notice from their models.
- Describe properties of multiplication they find in their models.
  - The order of the factors does not change the product.
  - When I multiply a number by 1, I get that number.
  - When I add zero to a number, I get the number I started with.
  - When I multiply a number times 0, the product is 0.
  - When I have 3 or more factors, I can change the order and grouping of the factors to make the problem easier to solve.
  - I can decompose one factor into 2 parts and multiply each part by the other factor and find the sum of those parts to help me find the product.
- Explain their reasoning to others.
- Write equations for the examples they have modeled.
- Solve problems that use these properties.

Addressing Student Misconceptions and Common Errors

Students often confuse multiplying by zero with adding to zero. Although this property seems obvious, providing students with problems and using models will help to reinforce the correct understanding.

The distributive property forms the foundation for all future work with multiplying whole numbers. However, in Grade 3, students should use this valuable property to help learn more difficult basic facts through array models. Introduce and continue work with this property using models throughout early work with multiplication. Students need opportunities to use and describe this property in order to make sense of it.

Notes

STANDARD 6 (3.OA.B.6)

Understand division as an unknown-factor problem. For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.

This Standard is an extension of previous work relating multiplication and division (3.OA.A.3 and 3.OA.A.4). It also supports the problem situations in Table 2 (page 256), as students solve problems that involve looking for missing factors they relate that work to multiplication. Thinking about division in terms of multiplication will help students to use the multiplication facts they know to become fluent with division facts. This Standard supports the use of fact families to reinforce the relationship between multiplication and division.

$6 \times 4 = 24$ $\quad 4 \times 6 = 24$ $\quad 24 \div 6 = 4$ $\quad 24 \div 4 = 6$ $\quad 6\overline{)24}$ (4)

$24 = 6 \times 4$ $\quad 24 = 4 \times 6$ $\quad 4 = 24 \div 6$ $\quad 6 = 24 \div 4$ $\quad 4\overline{)24}$ (6)

What the TEACHER does:

- Provide problem situations that involve finding a missing factor for students to model and solve. (See Table 2, page 256.)
- Make explicit connections between the model, the written multiplication equation, and the related division equation.
- Include class activities that relate division to thinking about a missing factor.

What the STUDENTS do:

- Describe information in problem situations and relate that information to written multiplication and division equations.
- Practice using missing factors to find the solution to the division problem.
  - To solve 42 ÷ 6, think what number multiplied by 6 equals 42.
  - Describe their thinking using words and numbers.

Addressing Student Misconceptions and Common Errors

Students often consider multiplication and division as discrete operations and do not understand the importance of the relationship between them as they learn basic facts or solve problems. It is important for students to understand division in terms of finding a missing factor and relate this work to writing division expressions and equations. Students need much experience identifying what information is known and what they are looking for using concrete materials and drawing pictures as well as asking themselves the right question, such as "How many groups of 7 can I make from 28?" Relating work with models to written missing factor multiplication equations and division equations is essential for students to develop this understanding.

Notes

Operations and Algebraic Thinking
3.OA.C*

Cluster C

Multiply and divide within 100.

STANDARD 7

3.OA.C.7: Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

*Major cluster

Operations and Algebraic Thinking 3.OA.C

Cluster C: Multiply and divide within 100.
Grade 3 Overview

Thinking about this cluster in terms of rigor, students build conceptual understanding of multiplication and division by modeling a variety of problem situations and using properties to develop reasoning strategies. From these experiences, fluency with facts becomes a natural progression that is much more effective than solely drilling multiplication and division tables. Although some students will need more practice than others, drilling facts should happen only after students have had substantive experiences building conceptual understanding through modeling and solving problems that reinforce the meaning of multiplication and division.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Through solving a variety of problems, students make sense of the meaning of multiplication and its relationship to division by identifying the number of groups and numbers of items in a group. They explain their reasoning and any strategies they use to solve problems and describe the relationship between multiplication and division. They use a variety of strategies that make sense to them to learn the basic multiplication and division facts and share these strategies with classmates. Students use various tools and models, including equal set models and arrays, to determine the product or the missing factor. Knowing and using appropriate vocabulary (*factor*, *missing factor*, *product*, *quotient*) will help students to associate meaning to their work and describe their thinking accurately to others. Exploring the properties and discussing what they notice helps to lay the foundation for the structure of multiplication and division not only within 100 but in future work as well.

Related Content Standards

4.NBT.B.4 4.NBT.B.5 4.NBT.B 6 5.NBT.B.5 5.NBT.B.6 5.NBT.B.7

Notes

STANDARD 7 (3.OA.C.7)

Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

This culminating Standard is the outcome of multiplication and division activities from all of the previous Standards in this domain. Students begin by modeling multiplication and division situations using concrete models, pictorial representations, and number lines. They solve a variety of multiplication and division problems focusing on the meaning of numbers in the situation and identifying what they know (number of groups, number of items in a group, or total number and use that information to figure out what they do not know. They explore and use properties of multiplication. They progress to writing equations for these situations and then relating division to multiplication facts by thinking of division in terms of multiplication (that is, to determine the quotient of 45 ÷ 9, they think "What number times 9 gives 45?"). All of these Standards lead to fluently multiplying and dividing within 100.

Students should have experience with multiplication and division equations written in all forms.

3×5 $\quad 3 \times \square = 15$ $\quad 15 \div 3$ $\quad 3\overline{)15}$

Strategies that students may use to become fluent with these facts include:

- Doubling (multiplication by 2)
- Multiplying by one (identity element)
- Multiplying by five (counting by 5)
- Multiplying by ten
- Doubling doubles (multiplication by 4)
- Square numbers (2 × 2, 3 × 3, 4 × 4, etc.)
- Composing and decomposing factors to use known facts (distributive property)
- Using the commutative property
- Multiplying by nine as related to multiplication by ten
- Building fact families
- Finding missing factors

What the TEACHER does:

- Ensure that students have many opportunities in many contexts (problem solving, using concrete materials and pictorial representations, using properties) to work with multiplication and division facts.
- Provide students with activities to make explicit connections between multiplication and division followed by classroom conversations and asking purposeful questions.
- Provide experiences that elicit student strategies to learning basic facts.
- Use drill and practice after students have demonstrated conceptual understanding and have strategies for a group of facts.

What the STUDENTS do:

- Solve problems and model examples that represent multiplication and division facts.
- Relate models to written equations.
- Develop understanding of the relationship between multiplication and division by identifying information and using that information to ask themselves questions that support understanding.
- Use strategies based on properties and patterns of multiplication to learn multiplication facts.
- Use multiplication facts in terms of a missing factor to learn division facts.

Addressing Student Misconceptions and Common Errors

The development of conceptual understanding must precede drill and practice exercises. Students who struggle with facts need more experience with concrete and pictorial representations, including describing what their models represent to make connections to basic facts. They need time and experience with developing strategies that are based on patterns and properties to help support learning their facts. It is important to give students time to learn and understand these concepts before procedural skill practice takes place.

Operations and Algebraic Thinking
3.OA.D*

Cluster D

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

STANDARD 8

3.OA.D.8: Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]

[3] This Standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

STANDARD 9

3.OA.D.9: Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

*Major cluster

Operations and Algebraic Thinking 3.OA.D

Cluster D: Solve problems involving the four operations, and identify and explain patterns in arithmetic.
Grade 3 Overview

Third graders have many experiences solving multiplication and division problems to build conceptual understanding of those operations. These problems should be extended to situations in which students will use an operation or combinations of operations to solve two-step problems using various models and representations. Estimation strategies not only help to extend conceptual understanding, but also help students think about the numbers in a problem and whether a solution is reasonable. This cluster also includes giving students opportunities to examine patterns in multiplication and division and how those patterns relate to the properties (and vice versa).

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students at this level use quantitative reasoning to solve single- and multi-step problems that include all four operations using models, pictures, words, and numbers. Students think about solutions in terms of reasonableness, asking themselves "Does this make sense?" Estimation strategies help to extend conceptual understanding, and to think about the numbers in a problem to determine if a solution is reasonable. Students explain their thinking using materials, pictures, words, and numbers. They listen to the reasoning of others and look for similarities and differences in various strategies used to solve a problem. Using a variety of representations and models helps students to solve problems and to deepen their understanding of the meaning of the operations. They begin to develop problem solving strategies, including make a model, draw a picture, make an organized list, and find a pattern. Students select appropriate tools, including concrete materials, graph paper, and pictures to help solve problems. They also ask themselves if a task can be completed by mental computation, estimation, or paper and pencil. For more complex situations, they might use a calculator. Using appropriate mathematical vocabulary and accurate units of measure are areas of focus as students begin to solve more sophisticated problems. They look for and extend mathematical patterns in a variety of situations, including tables and problems, and connect those patterns to the properties. These patterns help students to understand the structure of the four operations and should also be connected to the work in the Number and Operations in Base Ten (NBT) domain.

Related Content Standards

2.OA.A.1	2.MD.B.5	2.MD.C.8	3.MD.A.1	3.MD.A.2	3.MD.B.3	3.MC.C.7	3.G.A.1	4.0A.A.2
4.OA.A3	4.OA.C.5	4.MD.A.2	4.MD.B.4	4.MD.C.7				

STANDARD 8 (3.OA.D.8)

Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]

[3] This Standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

This Standard includes several connected mathematical ideas. Students solve two-step problems that include more than one operation by representing the information using concrete models, pictures including bar models, and number lines. Writing equations begins by making connections between the representations and the symbolic notation (equations). Although order of operations is not formally taught in Grade 3, thinking about the order in which operations should be a part of determining if answers are reasonable.

For example, consider the following problem:

Marcos bought a candy bar for 75¢ and 3 apples for 65¢ each. How much did Marcos spend?

Students need to realize that they need to find the total cost of the apples by multiplying before they add the cost of the candy bar. This begins informal thinking about order of operations (multiplication precedes addition and subtraction). Any other process would not result in a sensible solution. Explicit opportunities for discussions about possible models, strategies, and results should be an important part of this Standard.

Determining whether answers are reasonable by using number sense, understanding the context, the meaning of operations using mental computation strategies, and estimation strategies cannot be overemphasized as students work with all of the ideas imbedded in this Standard. Do not rush students through this process. It may be that one or two good problems will take a full mathematics class; assigning one or two follow-up problems for homework will encourage students to be thoughtful about their work and solution.

Using a letter standing for the unknown quantity should explicitly connect to previous work with identifying missing information that was represented by a box, underscore, or other symbols.

What the TEACHER does:

- Scaffold problems that use all four operations, including two-step problems that call for the use of different operations.
- Support students in interpreting problems, including identifying given, needed, and wanted information.
- Expect students to explain their solution strategies and to justify why their solution makes sense.

What the STUDENTS do:

- Solve problems using models, pictures, words, and numbers.
- Use a variety of problem solving strategies, including restating the problem in their own words, making models, and drawing pictures to represent their thinking.
- Explain how they solved the problem using accurate mathematical vocabulary and why their answer makes sense.

Addressing Student Misconceptions and Common Errors

Students who struggle with knowing what to do to solve problems will find it helpful to restate the problem in their own words. They should identify and underline the important information in the problem and determine what other information they might need in order to solve the problem. When they explain what the problem is asking, students will find that it will help them determine whether their answer is reasonable. Students who become easily frustrated with word problems may need carefully constructed questions to help direct them in determining what to do to solve the problem, but they should never be told what to do to reach a solution.

Notes

STANDARD 9 (3.OA.D.9)

Identify arithmetic patterns (including patterns in the addition table or multiplication table),and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.

The ability to recognize mathematical patterns is one of the most important characteristics of successful mathematics students. Mathematical ideas and concepts build on patterns, and the sooner students begin to recognize and identify patterns, the more likely mathematics will make sense to them. Identifying and explaining patterns leads students to develop the ability to make generalizations, which is the foundation of algebraic reasoning and more formal mathematical thinking.

Students should have many opportunities to explore, recognize, and talk about patterns in all of their mathematical work, but especially in addition and subtraction and the related operations of multiplication and division. Expecting students to justify their thinking is an important part of the discussion. "How do you know?" and "Does this always work?" are important questions for students to consider when describing and explaining patterns they have found.

What the TEACHER does:

- Provide students with copies of the addition or multiplication table of facts.
- Ask students to find any patterns they notice. They might color in the patterns and describe the numerical and visual pattern. Student-generated patterns are more meaningful than patterns they are shown.

What the STUDENTS do:

- Look for patterns on the addition or multiplication tables and color them. For example, if students shade all of the even numbers on the multiplication table, they will see that all of the numbers in the second, fourth, sixth, eighth, and tenth columns are shaded.
- Describe the patterns they have found.

Addressing Student Misconceptions and Common Errors

Students who have difficulty finding and describing patterns should start with simple examples and build to more complex patterns. They are more likely to notice visual patterns before numerical patterns. If possible, project the multiplication or additional tables and shaded patterns for all to see. This will help struggling students to visualize the patterns and then look at characteristics of the numbers.

Notes

Sample PLANNING PAGE

3.OA.A

Operations and Algebraic Thinking

Cluster A: Represent and solve problems involving multiplication and division.

Standard: 3.OA.A.2. *Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.* For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.

Standards for Mathematical Practice:

SFMP 1. Make sense of problems and persevere in solving them.

Students solve division problems using both meanings of division.

SFMP 3. Construct viable arguments and critique the reasoning of others.

They explain their reasoning using pictures, words, and numbers.

SFMP 4. Model with mathematics.

Square tiles and grip paper are used to model the problems.

SFMP 7. Look for and make use of structure.

Students develop a deep understanding of the partitive and measurement meaning of division.

Goal:

Students will experience and model situations involving partitive and measurement meanings of division. They make connections to previous experiences with the meaning of multiplication.

Planning:

Materials: Square tiles or similar counters, grid paper

Sample Activity:

Students work in groups of 3 to solve each of the following problems using the tiles. They draw their representations on the grid paper and discuss how the problems are similar and how they are different.

- Anita has 42 M&M'S to put in 6 cups. If she wants to put the same number of M&M'S in each cup, how many will she put in each?
- Anita has 42 M&M'S and she wants to put 7 in a cup. How many cups will she need?

Provide similar problems for students to solve.

Notes

Sample PLANNING PAGE (Continued)

3.OA.A

Questions/Prompts:

For each problem ask students what they know. Connect to previous experience with multiplication by asking questions such as:

- What number tells you the total number of M&M'S?
- What number tells you how many in a group?
- What number tells you the number of groups?

After students successfully model one set of problems, as they are working on additional problems ask them to them identify what the numbers in the problem represent (total, number of groups, or number of items in a group) and what they are looking for (number of groups or number of items in a group).

Ask students to write the equation using missing factor notation ($6 \times \_\_\_\_ = 42$) and then relate that to writing a division equation ($42 \div 6 = \_\_\_\_$).

Differentiating Instruction:

Struggling Students: Provide students with many opportunities to model the problems. Have physical representations for the sets (for example, paper cups) and allow them to act out the problem. Your question will need to be very explicit so they can identify whether they know the number of groups or the number of items in a group.

Use problems with simpler numbers for struggling students.

Extension: Once students have solved several sets of problems, those who easily identify the information in the problem and make connections between previous work with multiplication and this work with division can begin to solve problems that involve remainders.

Notes

PLANNING PAGE

3.OA.A

Operations and Algebraic Thinking
Cluster A: Represent and solve problems involving multiplication and division.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

PLANNING PAGE

3.OA.B

Operations and Algebraic Thinking

Cluster B: Understand properties of multiplication and the relationship between multiplication and division.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

PLANNING PAGE

3.OA.C

Operations and Algebraic Thinking
Cluster C: Multiply and divide within 100.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

PLANNING PAGE

3.OA.D

Operations and Algebraic Thinking

Cluster D: Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

Operations and Algebraic Thinking
4.OA.A*

Cluster A

Use the four operations with whole numbers to solve problems.

STANDARD 1 **4.OA.A.1:** Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

STANDARD 2 **4.OA.A.2:** Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[1]

[1] See Table 2 in the Resources, page 256.

STANDARD 3 **4.OA.A.3:** Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

*Major cluster

Operations and Algebraic Thinking 4.OA.A

Cluster A: Use the four operations with whole numbers to solve problems.
Grade 4 Overview

Fourth graders have worked with equal group and array/area problem situations for multiplication and division in Grade 3. Multiplication and division comparison situations are introduced in Grade 4. Students continue to work with one- and two-step problems that use all four operations, including problems in which remainders must be interpreted in terms of the question being asked in the problem.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students at this level use quantitative reasoning to solve single and multi-step problems that include all four operations using models, pictures, words, and numbers. In addition to equal group and area situations, they begin to solve multiplication and division comparison problems. They think about solutions in terms of reasonableness, asking themselves "Does this make sense?" Estimation strategies not only help to extend conceptual understanding but also students' thinking about the numbers in a problem to determine whether a solution is reasonable. Students explain their thinking using concrete materials, pictures, words, and numbers. They listen to the reasoning of others and look for similarities and differences in various strategies used to solve a problem. Using appropriate mathematical vocabulary and accurate units of measure are areas of focus as students begin to solve more sophisticated problems.

Students use various representations and models to help solve problems. They continue to develop problem solving strategies, including make a model, draw a picture, make an organized list, find a pattern, solve a simpler problem, and guess and check. Students select appropriate tools, including concrete materials, graph paper, and pictures to help solve problems. They also ask themselves whether a task can most efficiently completed by mental computation, estimation, or paper and pencil. For more complex situations, they might use a calculator.

Students look for and extend mathematical patterns in a variety of situations and connect those patterns to the properties. These patterns help students understand the structure of the four operations and should also be connected to the work in the Number and Operations in Base Ten (NBT) domain.

Related Content Standards

2.OA.A.1 2.MD.B.5 2.MD.C.8 3.MD.A.1 3.MD.A.2 3.MD.B.3 3.OA.D.8 3.MC.C.7 3.G.A.1

4.MD.A.2 4.MD.B.4 4.MD.C.7 5.MD.C.5 5.G.A.2

Notes

STANDARD 1 (4.OA.A.1)

Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

In the earlier grades students worked with additive comparisons. Megan has 25¢ and Liz has 5¢. How much more does Megan have than Liz? (2.OA.1). (What amount would be *added* to Liz's money to get Megan's amount?)

Fourth graders learn to compare these quantities multiplicatively. Megan has 5 times as much money as Liz. (What factor would *multiply* Liz's money to get Megan's amount?) A multiplicative comparison is a situation in which one quantity is described as a multiple of the other.

Another way to identify multiplicative comparisons is recognizing that in these situations there are two different sets being compared. The first set contains a certain number of items. The second set contains multiple copies of the first set. The language of multiplicative situations can be difficult for students. Students should become familiar with these and similar ways to describe multiplicative comparison situations.

Darlene has seven marbles. Danny has 3 times as many.

Danny has 3 times as many marbles as Darlene.

The number of marbles Danny has divided by 3 is the number of marbles Darlene has.

Danny has 3 times fewer marbles than Darlene.

This Standard should be taught in concert with 4.OA.A.2 so that students are consistently working among problem situations, models, and equations as they work with comparisons.

What the TEACHER does:

- Provide many opportunities for students to identify and model multiplicative comparison situations (Table 2, page 256).
- Reinforce appropriate vocabulary to describe comparison situations and appropriate examples for writing equations from these situations.

What the STUDENTS do:

- Read and interpret multiplicative comparison situations identifying which quantity is being multiplied and which factor is telling how many times.
- Write and identify equations and statements for multiplicative comparisons.

  5 × 3 Cathy has $5. Mary has three times as much. How much money does Mary have?

- Recognize different language that describes multiplicative comparisons. (See Table 2, page 256.)

Addressing Student Misconceptions and Common Errors

Students may struggle with applying their knowledge of multiplication and division facts to multiplicative situations since all of their previous experience was with equal groups and array models. They need many experiences connecting facts to the language of multiplicative comparisons. Using concrete models will support students in making this connection.

Notes

STANDARD 2 (4.OA.A.2)

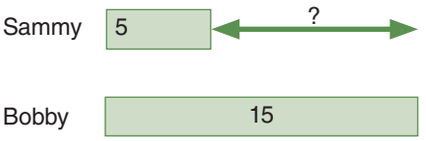
Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[1]

[1] See Table 2 in the Resources, page 256.

In previous grades students worked with tape diagrams and other models to show additive comparison situations (see Table 1, page 254). Similar models with different representations will help with understanding multiplicative comparison situations. Early problems should involve one step and then build to multiple-step problems that involve up to three steps.

Additive Comparison (5 + m = 15)

It takes Sammy 5 minutes to wash the dishes. It takes his brother Bobby 15 minutes to wash the dishes. How much longer does it take Bobby to wash the dishes?



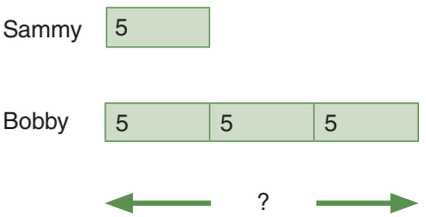
In this situation, we are finding the difference between Sammy's time and Bobby's time.

Multiplicative Comparisons

Product unknown: (3 × 5 = *t*)

It takes Sammy 5 minutes to wash the dishes. It takes his little brother Bobby 3 times as long. How long does it take Bobby to wash the dishes?

Think: 5 minutes 3 times would be?

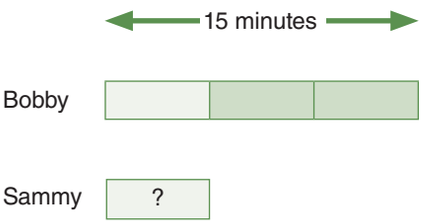


In this situation, Bobby's time is a multiple of Sammy's time.

Factor unknown (size of each group unknown) 3 × m = 15

It takes Bobby 15 minutes to wash the dishes. That is three times as long as it takes his brother Sammy. How long does it take Sammy to wash the dishes?

Think: Three groups make 15, how big is each group?



Factor unknown (number of groups unknown) $g \times 5 = 15$

It takes Sammy 5 minutes to wash the dishes and his little brother Bobby 15 minutes to wash the dishes. How many times as long does it take Sammy to wash the dishes?

Think: How many 5s to make 15?

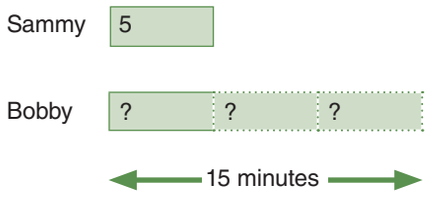


Table 2, page 256 in the Resources, provides various examples and models for comparison situations.

What the TEACHER does:

- Provide a variety of multiplicative comparisons (see Table 2, page 256) for students to model, describe, and solve.
- Facilitate explicit discussions in which students describe the information in the problem and use that information to represent the situation using models.
- Help students to make explicit connections between models (such as bar models), and written equations using both multiplication and division.
- Provide mixed additive and multiplicative comparison situations to help students distinguish between the two types of comparisons.
    - Additive comparisons focus on the difference between two quantities.
        - How many more?
        - How many less?
    - Multiplicative comparisons focus on comparing two quantities when one is a specified number of times greater or less than the given quantity.

What the STUDENTS do:

- Solve problems involving multiplicative comparisons using concrete materials, pictures, words, and numbers.
- Identify the information in the problem and how it relates to models.
- Write equations to represent the mathematics of the situation.

Addressing Student Misconceptions and Common Errors

Students may confuse additive and multiplicative situations. They need a variety of problems to model and discuss. Identifying what they know from the information in the problem and focusing on the question will help them to make sense of the problem. They should also consistently ask themselves if their answer makes sense.

Notes

STANDARD 3 (4.OA.A.3)

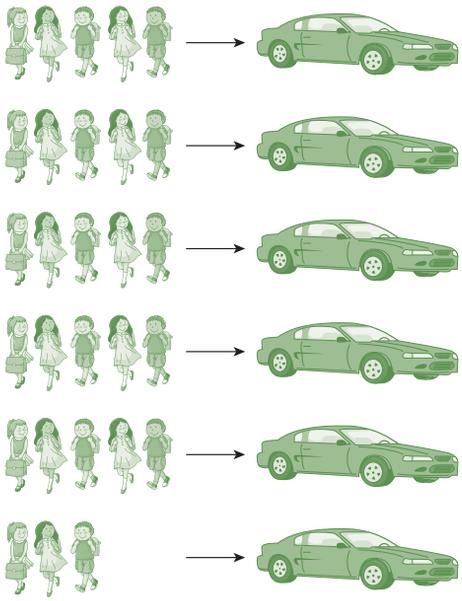
Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Students continue to work with problem situations as they build fluency with all four operations. Some students may still need to use concrete and pictorial models and connect these models to numerical equations representing the unknown with a letter before solving. Others may be able to determine the equation by recognizing the information in the problem and the question that is being asked.

Students should consistently use formal and informal estimation strategies to determine whether an answer is reasonable and fits the constraints of the given situation. They should have many opportunities to explain their strategies to predict a reasonable solution or to justify why they think their answer is reasonable. Remember, good discussions take time, so it is important to have rich conversations around a small number of good problems rather than superficial discussions about a large number of problems.

This will be students' first experience with interpreting remainders. Using models, identifying the information in the problem, focusing on the question, and discussing the reasonableness of results will help students determine what to do with a whole number remainder. (They will use remainders as fractions and decimals in later work.) It is important that students have ongoing experiences with a variety of problems in which they need to determine what to do with the remainder. Encouraging students to explain their reasoning in solving such problems will also help them to think about what to do with the remainder.

Problem	Model	Equation	What to do with the remainder
Eric bought a package of 18 batteries. Each toy robot uses 4 batteries. How many toy robots can be filled with batteries?		$18 \div 4 = 4$ remainder 2 That means that 4 robots can be filled with batteries and I will have 2 batteries left over.	The solution is the quotient. Drop the remainder.
Alyssa has a new bookcase with 4 shelves. Each shelf holds 9 books. If Alyssa had 38 books how many books will not fit in the bookcase?		$38 \div 4 = 9$ remainder 2 Two books will not fit on the shelves.	The remainder is the solution.

Problem	Model	Equation	What to do with the remainder
Twenty-eight students are going on the class picnic. Five students can ride in each car. How many cars will be needed for the trip?		28 ÷ 5 = 5 remainder 3 That means 5 cars can hold 25 students and another car will be needed for the extra 3 students. Six cars will be needed for the trip.	Add 1 to the quotient for the solution.

What the TEACHER does:

- Provide ongoing experiences with problems, including two- and three-step problems with all four operations.
- Facilitate small group and classroom discussions in which students show and explain their strategies and solution processes using materials or pictures, words, and numbers.
- Build on previous division problem experiences by including division problem situations that include remainders.
- Lead class discussions on what to do with the remainder by focusing on the problem question, the meaning of numbers in the problems and using models.

What the STUDENTS do:

- Solve multi-step problems with all four operations using models or pictures and numbers.
- Explain their problem solving processes and compare various ways of solving problems.
- In division situations with remainders, focus on the question asked to determine what to do with the remainder.
- Ask themselves if their solution makes sense.

Addressing Student Misconceptions and Common Errors

Students who struggle in determining what operation to use to solve a problem need additional experience understanding the operations in a variety of situations (see Tables 1 and 2, pages 254 and 256). They should have explicit practice with various problem solving strategies, including:

- Restating the problem in their own words.
- Identifying given, needed, and wanted information.
- Making a model or drawing a picture.
- Making a list.
- Acting it out.
- Finding a pattern.
- Writing an equation.
- Revisiting the question and asking themselves if the solution makes sense.

Students who do not have conceptual understanding or have focused work on division procedures tend to write answers to problems using the "r" notation. For example, 16 students are going canoeing. If each canoe holds 3 canoes, how many canoes will they need? The answer 5 r 1 makes no sense in this situation. Students need to focus on the question and reasonableness of solutions using strategies including models, pictures, and acting it out.

Operations and Algebraic Thinking
4.OA.B*

Cluster B

Gain familiarity with factors and multiples.

STANDARD 4

4.OA.B.4: Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

*Supporting cluster

Operations and Algebraic Thinking 4.OA.B

Cluster B: Gain familiarity with factors and multiples.
Grade 4 Overview

Students extend their understanding of multiplication and division to thinking about these operations in terms of composing and decomposing numbers into factors. For example, 12 can be decomposed into factors of 1, 2, 3, 4, 6, and 12 by knowing the multiplication facts that result in a product of 12. Making arrays will help students to build understanding of factors, reinforcing fluency with basic facts and extending to factor pairs beyond the basic facts. This Standard extends this understanding through recognizing prime numbers (numbers with exactly two factors) and composite numbers (number with more than two factors).

Standards for Mathematical Practice
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students extend their work with multiplication and division facts to focusing on finding factors and multiples of numbers less than 100. Although their facility with facts will help, using models and reasoning with the distributive property will help them find the factors that are not basic facts. For example, the factor pairs for 36 are 1×36, 2×18, 3×12, 4×9, and 6×6. Of these factor pairs, only 4×9 and 6×6 are considered basic facts.

Once students understand factors and multiples, they build on this understanding by defining and identifying prime and composite numbers. These concepts are important to future work with fractions. Students construct arguments based on the patterns they have found, including, for example, why 45 is a composite number and 47 is prime. They do this using models, words, and numbers.

Students extend their knowledge of basic facts by constructing arrays for numbers beyond the basic facts. For example, they can use a model of an 11×3 array to show that 3 and 11 are factors of 33 using graph paper, square tiles, and other appropriate tools. As they complete these activities, the vocabulary of multiplication is extended to include *factor*, *multiple*, *prime number*, and *composite number*. Students should be able to clearly define these words and use them in a variety of contexts.

Students recognize patterns as they explore numbers. Some numbers have exactly two factors and others have more than two factors. They will apply these structures and they begin to work with fractions. They use these patterns to make and justify generalizations such as "all even numbers other than 2 are composite because they will have more than two factors." or "All numbers other than 5 that end in 0 or 5 are composite because they will have 5 as a factor."

Related Content Standards

3.OA.B.6 3.NBT.A.3 4.NF.B.4.a 4.NF.B.4.b

STANDARD 4 (4.OA.B.4)

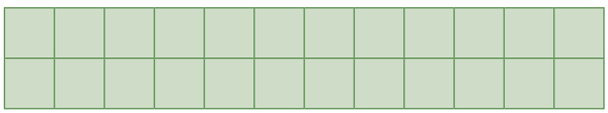
Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

This Standard builds on and extends students' knowledge of multiplication and division facts. Factor pairs include two numbers that when multiplied result in a particular product. The factor pairs of 28 include 1 × 28, 2 × 14, and 4 × 7, so the factors of 28 are 1, 2, 4, 7, and 28. Students can use square tiles to make arrays to find all of the factor pairs of a given number. They then explore patterns to build a conceptual understanding of prime numbers (numbers with exactly two factors) and composite numbers (numbers with more than two factors).

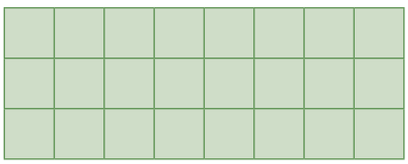
Multiples are the result of multiplying two whole numbers. Multiples can be related to factors, as shown below. Skip counting by a given number also results in the multiples of that number.

Using arrays to find the factors of 24.

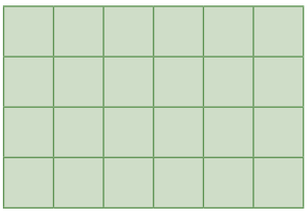
1 × 24



2 × 12



3 × 8



4 × 6

The *factor pairs* for 24 are 1 × 24, 2 × 12, 3 × 8, and 4 × 6.

The *factors* of 24 are 1, 2, 3, 4, 6, 12, and 24.

12 is a *multiple* of 1, 2, 3, 4, 6, and 12.

This Standard connects to 4.OA.C.5 in that it provides a variety of contexts for discussing number patterns that arise when students find factors and multiples. It is important to providing opportunities for students to use concrete materials, hundreds charts, and fact tables to discover patterns, followed by oral and written explanations describing those patterns, helps students develop a deeper understanding of factors, multiples, primes, and composites and to begin to make generalizations about the patterns they have found.

Such patterns include:

- Numbers that end in 0 have 10 as a factor. These numbers are multiples of 10.
- Numbers that end in 0 or 5 have 5 as a factor. These numbers are multiples of 5.
- Even numbers have 2 as a factor. These numbers are multiples of 2.
- Numbers that can be halved twice have 4 as a factor. These numbers are multiples of 4.

What the TEACHER does:

- Engage students in activities in which they use models such as arrays to find all of the factor pairs of a given number. Begin with a smaller range of numbers (1 to 20) and extend the range as students show understanding.
- Use games such as the factor game (http://illuminations.nctm.org/Activity.aspx?id=4134) to provide opportunities for students to find factors.
- As students make lists of factors, provide opportunities for them to discuss patterns.
- Use a variety of activities for students to explore finding multiples by skip counting and relate multiples to the products of a number.
- Have students connect their work with factors to identifying prime numbers (numbers with exactly two factors) and composite numbers (numbers with more than two factors). The multiples of a prime number are all composite numbers.
- Develop mathematical vocabulary including *factor*, *factor pair*, *and multiple*, *odd*, *even*, *prime*, and *composite*. Add these terms to the class mathematics word wall.

What the STUDENTS do:

- Students draw upon and extend their work with multiplication and division facts to determine the factors of a given number through a variety of activities.
- Discuss patterns they discover as they factor a number. (For example, all even numbers have 2 as a factor. Numbers that end in 0 or 5 have 5 as a factor.)
- List multiples of a given number using skip counting and other strategies.
- Identify and describe prime numbers as numbers that have exactly two factors.
- Identify and describe composite numbers as numbers that have more than two factors.

Addressing Student Misconceptions and Common Errors

Students often confuse the terms *factor* and *multiple*. Emphasizing the term *factor* as one of the numbers multiplied to get a product throughout all of the work with multiplication, and expecting students to use that term, should help avoid confusion. Telling students they multiply to get a *multiple* or defining *multiples* of a number as products of the number is also helpful. The more experience students have with these terms, the more accurate they will become when using them.

When listing multiples of a number, students may forget to include the number itself. Reminding students that multiples are the products of a number leads to a discussion of why a number is a factor and a multiple of itself, which is a result of the identity element of multiplication ($a \times 1 = a$).

Students may become confused about whether 1 is a prime or composite number, when actually it is neither prime nor composite because it has only one factor, itself. Developing precise definitions should help to eliminate this misconception.

Notes

Operations and Algebraic Thinking
4.OA.C*

Cluster C

Generate and analyze patterns.

STANDARD 5

4.OA.C.5: Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

*Additional cluster

Operations and Algebraic Thinking 4.OA.C

Cluster C: Generate and analyze patterns.
Grade 4 Overview

Finding, extending, generating, and describing patterns support developing conceptual understanding for all whole-number operations. Finding patterns is also an important strategy for solving problems. Students should have opportunities to extend and describe both physical patterns and numerical patterns.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students use problems as a context for finding and extending patterns. They reason about similarities and generate rules to describe numerical and geometric patterns. Students use models and tools to describe patterns they find in problems, in numbers, and in geometric figures and to extend these patterns to other situations. They develop lists of numbers given a rule and then describe any patterns in the list using appropriate vocabulary.

In finding patterns, students are developing a deeper understanding of the structure of all four operations and begin to make generalizations by constructing rules for their patterns.

Related Content Standards

3.OA.D.9 5.OA.B.3 5.NBT.A.2

Notes

STANDARD 5 (4.OA.C.5)

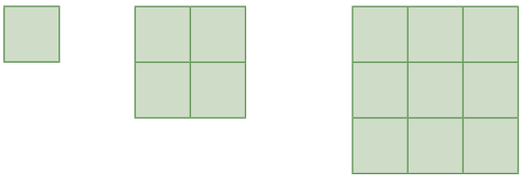
Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

Patterns that involve numbers or symbols can be repeating patterns or growing patterns. A repeating pattern is a cyclical repetition of an identifiable core. A linear growing pattern is a pattern that increases or decreases by a constant difference. Patterns can be shown by numeric or by geometric representations.

Number Pattern

Start with the number 4, add 5. 4, 9, 14, 19, 24, 29, . . . The numbers alternate with 4 and 9 in the ones place, which is an example of a repeating pattern. The number sequence increases by 5 which is a growing pattern.

Geometric Pattern



1 4 9

The shapes are all squares.

Each shape has one more tile in each row and column.

The numbers of tiles are multiples of factor pairs with the same factor.

$1 \times 1 = 1$ $2 \times 2 = 4$ $3 \times 3 = 9$

Students need a variety of opportunities in their regular mathematics work to create, extend, and describe patterns. Numerical patterns reinforce mastery of basic facts and understanding operations.

Given a geometric pattern or a numerical rule, students should extend the pattern and describe features of the pattern. They should have opportunities to describe what is happening with the pattern, but they do not need to generalize a particular rule. Describing physical patterns and connecting them to quantity supports recognition of more sophisticated patterns. Building patterns from problem situations and making an organized list are two invaluable problem solving strategies.

Finding a pattern is also an important strategy students can use to solve problems.

Example:

Anna has 5 pennies in her piggy bank. Each day she adds 6 more pennies. How many pennies will Anna have after 5 days?

Day	Pennies Added (equation)	Total Number of Pennies
0	$0 + 5$	5
1	$1 \times 6 + 5$	11
2	$2 \times 6 + 5$	17
3	$3 \times 6 + 5$	23
4	$4 \times 6 + 5$	29
5	$5 \times 6 + 5$	35

Student discuss what they notice and how they can continue the pattern for more days.

What the TEACHER does:

- Provide a variety of problems and models to give students opportunities to recognize, extend, and describe patterns.
- Help students to use a variety of strategies to organize their work so that patterns are more apparent. For example, making a list, extending and describing shapes, making a model will help students to recognize and describe patterns they see.
- Facilitate student discussions about patterns they find so that students become comfortable describing and writing their ideas using words and numbers.

What the STUDENTS do:

- Look for patterns in all of their mathematical work.
- Extend and describe patterns they find.
- When appropriate, make generalizations about patterns.

Addressing Student Misconceptions and Common Errors

Some students will have difficulty recognizing and extending patterns. Others may struggle to describe the pattern. It is important for students to have many experiences with patterns and to recognize those patterns. The more patterns are explored and related to mathematical concepts, the better students will become in recognizing and describing patterns.

Some students may "recognize" a pattern when there is no pattern. Asking students to describe and continue the pattern will help them to differentiate real patterns from non-patterns.

Notes

Sample PLANNING PAGE

4.OA.B

Operations and Algebraic Thinking

Cluster B: Gain familiarity with factors and multiples.

Standard: 4.OA.B.4. *Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.*

Standards for Mathematical Practice:

SFMP 1: Make sense of problems and persevere in solving them.
Students solve this problem to identify factors of given numbers.

SFMP 4: Model with mathematics.
Using a variety of rectangular arrays will help students find factors.

SFMP 6: Attend to precision.
Students work to find all of the possible solutions and use the terms *dimensions* and *factors* in relationship to their models.

SFMP 7: Look for and make use of structure.
Students use physical models to see the structure of multiplication and in a later lesson connect their findings to identifying prime and composite numbers.

Goal:

Students use rectangular arrays to solve a problem and connect their findings to identifying the factors of a given number.

Planning:

Materials: Square tiles, grip paper, problem page, recording sheet

Sample Activity:

- Introduce students to making rectangular arrays from a given number of square tiles.
- Introduce the term *dimensions*.
- Present students with the problem and allow them time to build all the arrays they can using the tiles, draw the arrays on grid paper, and then list the dimensions of the rectangles on the recording sheet.
- Connect the dimensions they have listed to the factors of the number. Then list the factors from least to greatest.
- Discussion follows on the number of factors and the number of boxes and how they are connected.

Notes

Questions/Prompts:

Some students may see the connection between the dimensions of the boxes and the factors and not want to use the materials. Encourage them to use the materials so they can see other patterns in their work and find the "non-basic fact" factors. (For example, 36 has factors of 3 and 12 and 2 and 18.)

Following student work time, be sure to take time to discuss patterns students have found. Questions similar to the following will facilitate the conversation:

- What did you notice about the even numbers?
- Was the same thing true for all of the odd numbers?
- Look at 4, 9, and 16. What do you notice about the number of boxes you found?
- What is different about the number 1?
- What other patterns did you notice?

Differentiating Instruction:

Struggling Students: Struggling students may not have time to work all the way to 40. However, you may find that they enjoy the activity and may want to continue on their own time. Make adjustments accordingly.

It is important that struggling students—especially those who have not mastered their facts—use the concrete materials to complete the task. They also may need support in finding all of the possible rectangles. Asking questions such as "*Can you make a box that is a 4 by something?*" will encourage them to keep trying.

Extension: Check to be sure students who complete the activity early have found *all* of the possible boxes. They can work beyond 40 and may enjoy the challenge of working with larger numbers.

Notes

Sample PLANNING PAGE

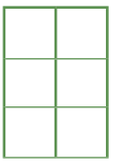
4.OA.B

Yummy Doggy Treats are square in shape. They are sold in rectangular boxes with one layer of treats in a box.

A box of six treats could be any of the following shapes.

1 by 6

6 by 1



____ by ____

____ by ____

The numbers under each box show the *dimensions* of the box. The first box on the left is 1 treat wide and 6 treats high or 1 by 6. The second box is 6 treats wide and 1 treat high. Write the dimensions of the other two boxes.

Our class has been hired to find all of the possible boxes that could be made for any number of Yummy treats from 1 to 40. Work in your groups to complete the table of treats. Look for patterns as you complete the table.

Notes

Table of Treats

Number of Treats	Dimensions of Boxes	Number of Boxes	Factors
6	1×6, 6×1, 2×3, 3×2	4	1, 2, 3, 6

PLANNING PAGE

4.OA.A

Operations and Algebraic Thinking

Cluster A: Use the four operations with whole numbers to solve problems.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

PLANNING PAGE

4.OA.B

Operations and Algebraic Thinking
Cluster B: Gain familiarity with factors and multiples.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

PLANNING PAGE

4.OA.C

Operations and Algebraic Thinking
Cluster C: Generate and analyze patterns.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

Operations and Algebraic Thinking
5.OA.A*

Cluster A

Write and interpret numerical expressions.

STANDARD 1 **5.OA.A.1:** Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

STANDARD 2 **5.OA.A.2:** Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.*

*Additional cluster

Operations and Algebraic Thinking 5.OA.A

Cluster A: Write and interpret numerical expressions.
Grade 5 Overview

In Grade 5, students continue to explore and work with numerical expressions in preparation for the Expressions and Equations domain coming in middle school. Students worked informally with order of operations in grades 3 and 4 as they solved multi-step problems through modeling and writing equations. According to the standards progression document (http://ime.math.arizona.edu/progressions/), this work should be exploratory, and expressions need not include nesting symbols.

Standards for Mathematical Practice
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

As students apply rules for order of operations they should explain their reasoning to others. They use scientific calculators (those that follow order of operations) to explore order of operations. Students use appropriate vocabulary to describe their work with grouping symbols and order of operations. As students explore order of operations and apply the rules in a variety of situations, they look for patterns and the structure of what is happening. They understand and apply calculating all multiplications and divisions before additions and subtractions within an expression. They make generalizations about the order of operations and grouping symbols and apply these rules to writing and solving expressions that include more than one operation and or grouping symbols.

Related Content Standards

6.EE.A.1 6.EE.A.2

Notes

STANDARD 1 (5.OA.A.1)

Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

Despite the traditional dependence on mnemonic phrases ("Please Excuse My Dear Aunt Sally") and mathematical convention, exploring the precedence of operations and the use of parentheses in expressions by solving a variety of multi-step problems allows students to reason about the order in which operations need to be performed. Explaining the order in which to calculate and knowing when to use parentheses can become confusing to students. Whenever possible, provide situations that model order of operations and make connections to the properties of addition and multiplication (associative and distributive).

Students need to understand that when looking at expressions, any operations in parentheses are completed first. Exponents, braces, and brackets will come later as described in the progressions document (http://ime.math.arizona.edu/progressions/).

What the TEACHER does:

- Introduce problem situations that show that multiplication and division are completed before addition and subtraction. Write and solve and equation for this problem.

  Mary bought a purse for $25 and 5 pairs of socks that cost $4 each. How much did she spend?

  $25 + (5 \times 4) = 45$

- Ask students if their answer makes sense.
- Provide students with experiences to solve equations that include parentheses with all four operations and explain their thinking.

  $5 \times (3 + 4) = 5 \times 7 = 35$ $\quad$ $18 - (4 + 3) = 18 - 7 = 11$
  $3 + (12 \div 4) = 3 + 3 = 6$ $\quad$ $(2 \times 5) + (3 \times 2) = 10 + 6 = 16$

- Provide students with equations and ask them to insert parentheses to make them true.

incorrect	$8 \div 4 \div 2 = 4$	$5 + 3 \times 7 = 56$
correct	$8 \div (4 \div 2) = 4$	$(5 + 3) \times 7 = 56$
incorrect	$4 + 8 - 4 \times 3 = 0$	$3 \times 3 + 6 \div 9 = 3$
correct	$(4 + 8) - (4 \times 3) = 0$	$3 \times (3 + 6) \div 9 = 3$

- Explain that multiplication and division are calculated in order from left to right before any addition or subtraction calculations. Have students use scientific calculators to solve equations with two to four operations. Students explain how the calculator solved the problem. This introduces students to the idea of order of operations, with multiplication or division being calculated from left to right and then additions and subtractions being performed.

  For example, when solving $4 + 6 \times 4$ the calculator will give an answer of 28, although most students will expect the answer to be 40. Students should reason that because multiplication is done before addition, 6×4 is calculated and then the 4 is added. $24 + 4 = 28$

- Students should work in small groups to solve equations with up to 4 operations by applying order of operations. They explain their solutions by describing how they solved the equations.

  $12 \div 4 + 3 \times 6$ Multiplications and divisions have equal weight, so $12 \div 4$ is calculated and then 3×6. The final answer is found by adding $3 + 18 = 21$.

  $16 - 8 \div 2 \times 4$ Divide 8 by 2 and multiply the result by 4. Subtract the result from 16.

  $16 - 16 = 0$

  $18 - 14 \div 2 + 5 \times 2$ Divide 14 by 2 and then multiply 5×2. Think $18 - 7 + 10 = 21$.

What the STUDENTS do:

- Solve problems and equations that include parentheses.
- Solve problems and equations that employ order of operations.
- Explain their thinking as they use order of operations to solve a variety of examples.

Addressing Student Misconceptions and Common Errors

Some students will have the misconception that all multiplications are calculated before divisions and additions are calculated before subtractions. Scaffold examples for students to practice solving multiplications and/or divisions in order from left to right and then additions and/or subtractions in order from left to right. Although parentheses are not necessary when the equation is written accurately, some students will find it helpful to add grouping symbols in order to solve equations and word problems.

STANDARD 2 (5.OA.A.2)

Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. For example, express the calculation "add 8 and 7, then multiply by 2" as $2 \times (8 + 7)$. Recognize that $3 \times (18932 + 921)$ is three times as large as $18932 + 921$, without having to calculate the indicated sum or product.

This Standard extends the work of the previous Standard by having students write and interpret numerical expressions. Moving from expressions to words and from words to expressions will reinforce understanding of order of operations.

What the TEACHER does:

- After students have had experiences solving many examples following order of operations, offer opportunities to play target number games in which they have to write equations using order of operations to make a target number and explain their reasoning (http://www.mathwire.com/games/numbsensegames.html, http://illuminations.nctm.org/Lesson.aspx?id=2962).
- Encourage games such as "I have who has" with an expression written in numbers and written in words to give students additional opportunities to understand how to read and interpret expressions without evaluating them (finding the solution).

What the STUDENTS do:

- Given a mathematical expression in words, write the numerical expression.

  Three times six added to seven would be written $7 + 3 \times 6$

- Given a numerical expression, translate it into words.

  $12 - (14 \div 7)$ could be read as *the quotient of 14 and 7 subtracted from 12.*

GRADE 5

Addressing Student Misconceptions and Common Errors

Struggling students should begin with simple expressions. Because there may be several ways to write or read an expression, they should justify how they arrived at their answers.

Notes

Operations and Algebraic Thinking
5.OA.B*

Cluster B

Analyze patterns and relationships.

STANDARD 3

5.OA.B.3: Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

*Additional cluster

Operations and Algebraic Thinking 5.OA.B

Cluster B: Analyze patterns and relationships.
Grade 5 Overview

Students extend the work of the previous cluster to graphing ordered pairs on a one quadrant coordinate plane and discussing the visual patterns on the graph.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students use problems as a context for finding and extending patterns. They reason to find similarities and determine rules to identify numerical and geometric patterns and describe patterns they find in problems and in numbers and in geometric figures. They compare descriptions and look for counterexamples. Students use models and tools to help extend and describe shape patterns. Given a rule, students develop lists of numbers and describe any patterns in the list using appropriate vocabulary. In finding patterns, students are developing a deeper understanding of the structure of all four operations and begin to make generalizations by constructing rules for their patterns.

Related Content Standards

3.OA.D.9 4.OA.C.5

Notes

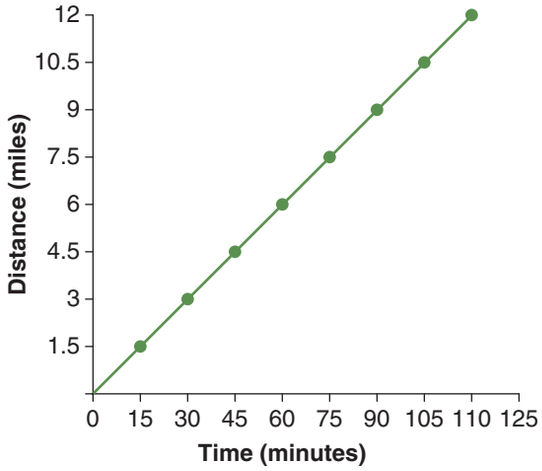
STANDARD 3 (5.OA.B.3)

Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.

This Standard relates to previous work in Grade 4 with patterns and includes several skills beyond solving the problem and identifying a pattern. Students need experience identifying, recording, and graphing ordered pairs on a coordinate plane (positive numbers only). In solving problems, they begin by making a list to find a solution and then graph the ordered pairs in that list. It is appropriate to begin with graphing a single rule, then discuss and describe the graph and how it relates to the problem situation. Proceed with situations in which two rules are generated. After graphing the ordered pairs for each rule, students analyze and discuss the relationship between the two results.

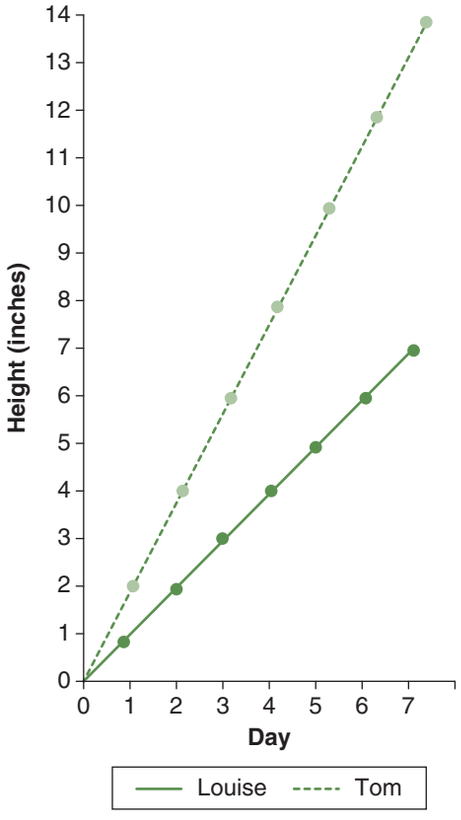
Example 1:

Berto is going on a bike trip. If he travels 1.5 miles every 15 minutes, how far will he ride in 2 hours?



Example 2:

Louise and Tom are working on a science project. They both plant seeds and measure the height of the plant every day. Louise's plant grows 1 inch a day and Tom's plant grows 2 inches each day. Draw a graph of their seeds' growth for the first week. Describe what you notice about each person's plant.



What the TEACHER does:

- Provide students with problems to solve in which students make a T-table and generate numerical patterns (see examples above), including problems with two rules or two patterns.
- Provide students with "mathematical rules" that give them additional opportunities to make a table and generate a sequence of numbers based on the rule.
  - Rule: Double the number and add 1.
  - Rule: Find $\frac{1}{2}$ of the number.
- Facilitate student discussions describing patterns in their tables.
- Support students as they list ordered pairs from the tables and plot the points on a coordinate grid.
- Facilitate student discussions in which they informally describe the visual patterns of their graphs (for example, "The graph is a line that starts at 0" or "The line goes up and to the right.")

What the STUDENTS do:

- Make a table to solve problems.
- Describe patterns.
- Plot pairs on a coordinate grid.
- Describe the graphs.

Addressing Student Misconceptions and Common Errors

Students often reverse coordinates when plotting them on a coordinate plane. They use the first number to count up the *y*-axis and then count over the second number in the ordered pair on the *x*-axis. Have students identify and label the *x*-axis and the *y*-axis on their coordinate planes. Remind them that since *x* is before *y* alphabetically, they move across the *x*-axis first and then up the *y*-axis. Students should also describe the difference of the location of (3,5) and (5,3) on the coordinate plane. Give students many opportunities to describe how to plot a point using a variety of ordered pairs.

Notes

Sample PLANNING PAGE

5.OA.A

Operations and Algebraic Thinking

Cluster A: Write and interpret numerical expressions.

Standard: 5.OA.A.1. *Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.*

Standards for Mathematical Practice:

SFMP 6. Attend to precision.

Students apply previous work with order of operations and pay close attention to the use of parentheses and the order of operations.

Goal:

Students will practice using order of operations to play a game in which they need to write expressions to make a given number.

Planning:

Materials: Dice or spinner with numbers 1 through 9, teacher-made cards with numbers from 15 to 30.

Sample Activity:

Students play in groups of four. Each person spins the spinner or rolls a die. They record the four numbers for that round. The teacher draws a card and gives the target number. Groups work together to make expressions using their four numbers following order of operations, including using parentheses, that will make the target number. They may use two, three, or all four of the numbers.

Students share their expressions and explain how they make the target using order of operations.

Sample: Playing numbers 2, 3, 3, 5 Target 16

$3 + 3 + (2 \times 5)$ or $3 + 3 + 2 \times 5$

Questions/Prompts:

As students share their expressions, you may find that as they explain what they did, it does not match how they wrote the equation. Ask questions to clarify student thinking, such as, "What did you do first? How did you write that in your equation?" You may also want to have them clarify why.

$3 + 3 + 2 \times 5$ is not equal to 40.

Differentiating Instruction:

Struggling Students: Work with a smaller range of numbers or use three numbers to make the target number.

Although parentheses may not be needed, students may find it easier to include parentheses in their expressions.

Extension: Students who easily find expressions should be encouraged to find more expressions. They can also write their expressions in different ways. For example, $3 + 3 + (2 \times 5)$ could also be written as

$2 \times 5 + 3 + 3$.

These students also work on being precise in their explanations of how their expressions are accurate.

PLANNING PAGE

5.OA.A

Operations and Algebraic Thinking
Cluster A: Write and interpret numerical expressions.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

PLANNING PAGE

5.OA.B

Operations and Algebraic Thinking
Cluster B: Analyze patterns and relationships.

Standard:

Standards for Mathematical Practice:

Goal:

Planning:

Materials:

Sample Activity:

Questions/Prompts:

Differentiating Instruction:

Struggling Students:

Extension:

Reflection Questions: Operations and Algebraic Thinking

1. How does the use of problem situations help students to develop a conceptual understanding of multiplication and division?

2. Discuss each problem situation in Table 2: Multiplication and Division Situations, Grades 3–5, in the Resources. As a group, develop a set of grade-level-appropriate problems you can use for each situation. What are some developmentally appropriate models for your grade level? Refer to Standards in this domain to determine the range of numbers with which students will be working at your grade level, and include fractions and mixed numbers.

3. Discuss strategies for introducing the problem types and how this progresses across grade levels so that you can build upon each other's work.

4. How do the properties of multiplication help students to develop a deeper understanding of multiplication and division as well as become fluent with procedural skills?

5. What are some activities you can use to help students "discover" the properties so they understand and use them? (Remember that students do not need to know a property's formal name at this level.)

6. How does the expectation for fluency with facts develop across grade levels in this domain? Talk about strategies that students can use to become fluent with facts using understanding rather than rote memorization.